Spin Geometry

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Contents

The subject of these lecture notes is *global analysis*, in particular the global analysis of the Dirac operator. We shall be concerned with the study of geometrically motivated linear PDEs on manifolds, which will then give us information on the geometry of the underlying manifold. The most important aspects of this subject include the following:

- In keeping with our global point of view, we shall consider PDEs on sections of *smooth vector bundles* on differentiable manifolds.
- The study of the (generalized) Dirac operator. For their construction we shall will need the concepts of *Clifford algebras* and *Spin structures*.
- The solution theory for linear PDEs is based on the theory of *Hilbert spaces*, in particular *Sobolev spaces*.
- Our main interest will be in elliptic PDEs, the Dirac operator being the most important example. For these operators we shall develop the theory of *elliptic regularity*. An important application is the *Hodge Theorem* on closed Riemannian manifolds, asserting that one can always find a unique harmonic representative in every de Rham cohomology class.
- We will study the spectrum of elliptic operators, which (over compact manifolds) consists only of eigenvalues. It is interesting to ask what geometric information is encoded in the spectrum, which is the topic of *spectral geometry*.
- The main goal of these lecture notes is to explain the *Atiyah-Singer Index Theorem*. It strikes a bridge between analytic properties of elliptic operators and global topological properties of the manifold in question, expressed through the *characteristic classes* of the tangent bundle.
- The proof of the Index Theorem presented in these lecture notes is based on the *heat equation* and the study of the asymptotics of the *heat kernel*.
- As applications we will discuss the *Signature Theorem* of Hirzebruch in differential topology, obstructions for the existence of *positive scalar curvature* metrics on spin manifolds and integrality theorems for characteristic numbers.

These lecture notes are based on the book *Elliptic operators, topology, and asymptotic methods* by John Roe (Pitman Research Notes in Mathematics **395**, 1998). In the following, we will make references to this book by '[Roe]'.

1 Vector bundles, Differential Forms, Connections, and Curvature

Good references for the material in this section include

Lawrence Conlon, *Differentiable manifolds*, Second Edition, Modern Birkhäuser Classics

(which can be downloaded via our library) as well as sections 8 and 9 of

John Milnor, *Morse theory*, Princeton University Press.

1.1 Sections of Vector Bundles

Let M be an n-dimensional smooth manifold and let $V \to M$ be a smooth real vector bundle. The real vector space of smooth functions on M will be denoted by $C^\infty(M)$. The $C^\infty(M)$-module of smooth sections $C^\infty(V)$ is the space of all smooth maps

$$Y\colon M \to V$$

with $Y(x) \in V_x$ (the fiber over $x \in M$) for all $x \in M$. Some authors also use the notation $\Gamma(V)$ to denote this space. If we let $\underline{\mathbb{R}} \to M$ denote the trivial vector bundle, we hence have $C^\infty(M) = C^\infty(\underline{\mathbb{R}})$. In other words, sections of vector bundles may be considered to be generalizations of functions.

1.2 Tangent Bundle

A particularly important vector bundle is the *tangent bundle* $TM \to M$ of a manifold. Sections of this bundle are called *smooth vector fields* on M. In local coordinates $(x^1, \ldots, x^n)$ on an open subset $U \subset M$, a vector field $X \in C^\infty(TM)$ takes the form

$$X|_U = \sum_{i=1}^n X^i \partial_i$$

with smooth coefficient functions X^i. Here, ∂_i denotes the directional derivative in the i-th coordinate direction. For a smooth function f on U and $x \in U$, we have $\partial_i(x)(f) = \frac{\partial f}{\partial x^i}(x)$. If $X \in C^\infty(TM)$ is a vector field and $f \in C^\infty(M)$ we define a function

$$\nabla_X f = Xf\colon M \to \mathbb{R}$$

as follows. On a coordinate neighborhood U we let

$$(Xf)(x) = \sum_{i=1}^n X^i \partial_i(x)(f) \qquad (x \in U).$$

It is easy to check that this definition is independent of the choice of local coordinates. Indeed, if $(y^1, \ldots, y^n)$ is another set of coordinates and if we let $\frac{\partial y^j}{\partial x^i}$ denote the Jacobian of the coordinate transform from x to y, then we have

$$\frac{\partial f}{\partial x^i} = \sum_{j=1}^n \frac{\partial y^j}{\partial x^i} \frac{\partial f}{\partial y^j}.$$

The expression Xf is also called the *Lie derivative* of f in direction of X. It is a *derivation* meaning that we have

$$X(fg) = Xf \cdot g + f \cdot Xg \qquad (f, g \in C^\infty(M)).$$

As shown in a course on differential geometry, the space of derivations of the algebra $C^\infty(M)$ may be identified with the space of vector fields on M.

1.3 Constructing New Vector Bundles from Old

We recall that any functorial (and smooth) construction of vector spaces (such as the dual space, direct sums, tensor products, homomorphism spaces, exterior and symmetric powers) may be applied fiber wise to vector bundles. Hence we have the direct sum $V_1 \oplus V_2 \to M$ of vector bundles $V_1, V_2 \to M$ and the dual vector bundle $V^* \to M$ with fiber V_x^* over $x \in M$. For another example, the fiber of $\mathrm{Hom}(V_1, V_2)$ over $x \in M$ is $\mathrm{Hom}\left((V_1)_x, (V_2)_x\right)$.

1.4 Differential Forms

An important example of this construction is the *cotangent bundle* T^*M of a manifold. Sections of T^*M are called *differential forms*. We write $\Omega^1(M) = C^\infty(T^*M)$. The basis dual to $\partial_1, \ldots, \partial_n$ will be denoted $dx^1, \ldots, dx^n$. More generally, we will use the notation $\Omega^m(M) = C^\infty(\Lambda^m T^*M)$. By choosing local coordinates on $U \subset M$ a differential 1-form $\omega \in \Omega^1(M)$ may be expressed as

$$\omega|_U = \sum_{1 \leq i \leq n} \omega_i dx^i$$

with smooth coefficient functions $\omega_i \in C^\infty(U)$.

There is a unique sequence of linear maps

$$d^m \colon \Omega^m(M) \to \Omega^{m+1}(M)$$

satisfying

- $df = \sum_{i=1}^n (\partial_i f) dx^i$
- $d^2 = 0$
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta$

Because $d^2 = 0$ we may define

$$H^m_{\mathrm{dR}}(M) = \frac{\ker(d^m)}{\mathrm{im}(d^{m-1})}.$$

This is the m-th *de Rham cohomology* of M. Together with the wedge product it is a graded commutative ring with unit, i.e. $[\omega] \wedge [\eta] = (-1)^{\deg(\omega)\cdot\deg(\eta)} [\eta] \wedge [\omega]$.

If M is a closed Riemannian manifold, we will prove later (see *Hodge theory*) that every cohomology class contains a unique harmonic representative. This means that $\Delta\omega = 0$ for the Hodge-Laplacian on m-forms (see also exercise 4 on worksheet 1). The Theorem of de Rham identifies the so defined de Rham cohomology with the singular cohomology of M.

1.5 Homomorphism bundles and Tensors

Suppose V, W are vector bundles on M and let $\phi \in C^\infty(\mathrm{Hom}(V, W))$ be a section of $V^* \otimes W$. Then we may define a linear map $\hat{\phi} \colon C^\infty(V) \to C^\infty(W)$ by

$$\hat{\phi}(X)(x) = \phi(x)(X(x)).$$

This assertion has a converse, which will be discussed on the exercise sheets:

Proposition 1. *Every $C^\infty(M)$-linear map $f \colon C^\infty(V) \to C^\infty(W)$ can be written as $f = \hat{\phi}$ for a unique $\phi \in C^\infty(\mathrm{Hom}(V, W))$.*

In other words: $f \colon C^\infty(V) \to C^\infty(W)$ being $C^\infty(M)$-linear is equivalent to $f(X)(x)$ being dependent only on $X(x)$ (so that f is 'tensorial').

Proof. The proof has two steps. First, we prove that $f(s)(p)$ depends only on $s|_U$ if $p \in U$. For this it suffices to show that $s|_U = 0$ implies $f(s)(p) = 0$. Let χ be a smooth function with $\chi(p) = 0$ and $\chi|_{M\setminus U} = 1$. Then $\chi s = s$, so

$$f(s)(p) = f(\chi s)(p) = \chi(p) f(s)(p) = 0.$$

In the second step, we show that $f(s)(p)$ depends only on the value $s(p)$. This may then be taken as definition for $\phi(p)(s(p))$ and it has the required property. For this, suppose $s(p) = 0$. We wish to prove $f(s)(p) = 0$. Let $(U, \phi = (x^1, \dots, x^n))$, be a chart neighborhood centered at p (i.e. $\phi(p) = 0$) in which $V = U \times \mathbb{R}^k$. Then s may be written in this local frame as

$$s|_U = \sum_{i=1}^k f^i e_i, \qquad f^i \in C^\infty(U).$$

By assumption, $f^i(p) = 0$. Using the first step, we get

$$f(s)(p) = f(s|_U)(p) = \sum_i f^i(p) f(e_i) = 0. \qquad \square$$

Introducing the notation $\Omega^m(V) = C^\infty(\Lambda^m T^*M \otimes V)$, we may therefore identify elements of $\Omega^1(V)$ with $C^\infty(M)$-linear maps $C^\infty(TM) \to C^\infty(V)$.

1.6 Covariant Derivatives

For $X \in C^\infty(M)$ we want to generalize the directional derivative Xf from functions f to tensor fields (e.g. differential forms). In a vector bundle V of rank k, every point $x \in M$ has an open neighborhood $U \subset M$ and a smooth vector bundle isomorphism

$$V|_U = U \times \mathbb{R}^k.$$

In this way, the standard basis $(e_1, \dots, e_k)$ of $\mathbb{R}^k$ defines local sections $e_1, \dots, e_k$ of $C^\infty(V|_U)$. At every point $x \in U$, the evaluations $e_1(x), \dots, e_k(x)$ define a basis of V_x. Therefore any section $Y \in C^\infty(V)$ may be expressed locally as

$$Y|_U = \sum_{i=1}^k Y^i e_i$$

for smooth coefficient functions $Y^i \colon U \to \mathbb{R}$. We call $(e_1, \dots, e_k)$ a local frame of V over U. If we choose local coordinates $(x^1, \dots, x^n)$ on U we could try to define

$$\nabla_X(Y) = \sum X^i (\partial_i Y^j) \partial_j.$$

However, this depends on the choice of trivialization of V and on the choice of local coordinates $(x^1, \dots, x^n)$.

In general, we make the following definition:

Definition 2. *A connection on V is an $\mathbb{R}$-linear map*

$$\nabla \colon C^\infty(TM) \otimes_{\mathbb{R}} C^\infty(V) \to C^\infty(V), (X, Y) \mapsto \nabla_X Y$$

satisfying the following properties for $f \in C^\infty(M)$:

1. $\nabla_{fX} Y = f \nabla_X Y$,
2. $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$.

We call $\nabla_X Y$ the covariant derivative of Y in direction of X. For fixed Y the map $X \mapsto \nabla_X Y$ is $C^\infty(M)$-linear. However, for fixed X the map $Y \mapsto \nabla_X Y$ is only $\mathbb{R}$-linear. On the exercise sheets we will see that the set of all connections is an affine space over $\Omega^1(\mathrm{End}(V))$ and that the value of $\nabla_X Y$ at $x \in M$ depends only on the value of Y in a neighborhood of x. Hence the covariant derivative is a *local operator*.

By choosing trivializations $V|_U = U \times \mathbb{R}^m$ and $TM|_U = U \times \mathbb{R}^n$ we may define functions $\Gamma_{ij}^k \colon U \to \mathbb{R}$ by the equations

$$\nabla_{\partial_i} e_j = \sum \Gamma_{ij}^k e_k.$$

(as above, the sections e_j correspond in the trivialization of $TM|_U$ to the standard basis of $\mathbb{R}^n$.) The functions Γ_{ij}^k are called the *Christoffel symbols* of ∇ (with respect to the given trivializations). Then the covariant derivative of $Y = Y^j e_j$ may be expressed as

$$\nabla_i(Y^j e_j) = (\partial_i Y^j) e_j + Y^j \Gamma_{ij}^k e_k. \tag{1}$$

Here we have used the notation $\nabla_i = \nabla_{e_i}$. By defining

$$e_j \mapsto \sum \Gamma_{ij}^k e_k$$

we get an element of $C^\infty(\mathrm{End}(V))$. We identify a section Y with a smooth map $(Y^1, \ldots, Y^m) \colon U \to \mathbb{R}^m$. Formula (1) may be restated as

$$\nabla_i Y = \partial_i Y + \Gamma_i \cdot Y$$

where $\Gamma_i = \Gamma_{ij}^k$ is viewed as an $(m \times m)$-matrix with values in $C^\infty(U)$, the row index being k and the column index being j. In this sense we have the equation

$$\nabla_i = \partial_i + \Gamma_i.$$

On the other hand, any n-tuple of sections $\Gamma_1, \ldots, \Gamma_n \in C^\infty(\mathrm{End}(V))$ defines a connection on $V|_U$ by setting $\nabla_i = \partial_i + \Gamma_i$.

1.7 Parallel transport

The geometric interpretation of a covariant derivative is that it gives a notion of *parallel transport*. Let $\gamma \colon [a,b] \to M$ be a smooth curve in M. A *section of V along γ* is a smooth map $X \colon [a,b] \to V$ with $X(t) \in V_{\gamma(t)}$ for all $t \in [a,b]$ (i.e. a section of the pullback of V along γ).

A covariant derivative on V also determines a covariant derivative ∇_t on sections along γ, uniquely characterized by the following conditions:

1. $\nabla_t(X+Y) = \nabla_t(X) + \nabla_t(Y)$
2. $\nabla_t(fX) = \frac{df}{dt} X + f \cdot \nabla_t X$ for smooth $f \colon [a,b] \to \mathbb{R}$
3. For a section Y of V and $X(t) = Y(\gamma(t))$ we have $\nabla_t X = \nabla_{\gamma'(t)} Y$.

This follows easily by choosing local coordinates $(x^1, \ldots, x^n)$ and a local frame $(e_1, \ldots, e_k)$ of V. Then a section Y along γ may be written as

$$Y = \sum Y^j e_j$$

for smooth $Y^j \colon [a,b] \to \mathbb{R}$. It is necessary and sufficient that

$$\nabla_t Y = \sum \frac{dY^i}{dt} e_i + Y^i \nabla_{\gamma'(t)} e_i$$

We have

$$\nabla_{\gamma'(t)} Y = \frac{d\gamma^i}{dt} \nabla_i Y = \frac{d\gamma^i}{dt}(\partial_i Y^j e_j + Y^j \nabla_i e_j) = \frac{dY^j}{dt} e_j + Y^j \nabla_{\gamma'(t)} e_j$$

This proves 3. from above.

Definition 3. *A section Y of V along γ is called* parallel *if*

$$\nabla_t Y = 0.$$

This equation is an ordinary linear differential equation of first order on the space of sections of V along γ on the coefficient functions $(Y^1, \dots, Y^k)$. For any given initial value $Y(0) \in T_{\gamma(0)} \cong \mathbb{R}^n$ the general theory of such equations gives us a unique solution of this differential equation. The resulting vector field Y is called the *parallel extension* of $Y(0)$ along γ. From the uniqueness of the solution of such equations we see that linearly independent choices of $Y(0)$ induce linearly independent parallel translates $Y(t)$ for all $t \in [a, b]$. Choosing a basis $(e_1, \dots, e_k)$ of V_x we therefore obtain a unique *parallel frame* $(e_1, \dots, e_k)$ of V along γ.

Suppose that $X \in T_x M$ and that $\gamma\colon (-\varepsilon, +\varepsilon) \to M$ is a curve with $\gamma'(0) = X$. Let $(e_1, \dots, e_k)$ be a parallel frame along γ. For a section Y of V we may write

$$Y(\gamma(t)) = Y^j(t) e_j$$

for smooth functions $Y^i\colon (-\varepsilon, +\varepsilon) \to \mathbb{R}$. For $X = X^i \partial_i$ we then have

$$\nabla_X Y = (\nabla_X Y^j) e_j = \sum X^i (\partial_i Y^j) e_j$$

which is similar to the naive formula for the connection from above.

Remark 4. *We have seen that a covariant derivatives induce a parallel transport along curves. In fact, the converse also holds.*

In general, the parallel transport depends strongly upon the curve γ. Thus if $\tilde{\gamma}$ is another curve from $\gamma(0)$ to $\gamma(1)$, the parallel transport along $\tilde{\gamma}$ is different from that along γ.

1.8 Riemannian metrics

Suppose we are given a fiber wise inner product $(-,-)$ on V. That is, let $(-,-)$ be a smooth map $V \oplus V \to \mathbb{R}$ that restricts on every fiber V_x, $x \in M$, to an inner product $(-,-)_x\colon V_x \times V_x \to \mathbb{R}$.

If $V|_U = U \times \mathbb{R}^k$ is a trivialization of V, then we get a map

$$U \to \mathbb{R}^{k \times k}$$

which associates to every $x \in U$ the matrix representing the inner product $(-,-)_x$ for the basis $(e_1, \dots, e_k)$ of $\mathbb{R}^k$. Using a partition of unity one sees that every vector bundle admits such a inner product. The set of all such inner products is convex. An inner product on the tangent bundle $TM \to M$ is called a *Riemannian metric.*

Definition 5. *A covariant derivative ∇ on V is* compatible *with the inner product $(-,-)$ if parallel transport along every curve $\gamma\colon [a, b] \to M$ preserves the inner product. That is, for all parallel sections Y_1, Y_2 of V we require that*

$$(Y_1(t), Y_2(t)) = (Y_1(a), Y_2(a)) \qquad \forall t \in [a, b].$$

Proposition 6. *A covariant derivative is compatible with $(-,-)$ precisely when*

$$\nabla_X (Y_1, Y_2) = (\nabla_X Y_1, Y_2) + (Y_1, \nabla_X Y_2)$$

for all $X \in C^\infty(TM)$, $Y_1, Y_2 \in C^\infty(V)$.

Recall here that we use the notation $\nabla_X f = X(f)$ for functions $f \in C^\infty(M)$.

Proof. See exercise sheet 1. $\square$

Theorem 7. *Let (M, g) be a Riemannian manifold. Then there is a unique connection ∇ on TM compatible with g which in addition is* symmetric *(or* torsion-free*). This means that*

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \forall X, Y \in C^\infty(TM).$$

This connection is called the Levi-Civita connection *on (M, g).*

Here, $[X, Y]$ is the *commutator* of vector fields X, Y. Identifying vector fields with derivations $C^\infty(M) \to C^\infty(M)$, the commutator is given by the derivation

$$[X, Y]f = X(Y(f)) - Y(X(f)).$$

In local coordinates,

$$\begin{aligned}(XY - YX)(f) &= X^i \partial_i (Y^j \partial_j f) - Y^j \partial_j (X^i \partial_i f) \\ &= X^i (\partial_i Y^j \partial_j f + Y^j \partial_i \partial_j f) - Y^j (\partial_j X^i \partial_i f + X^i \partial_j \partial_i f) \\ &= (X^i \partial_i Y^j \partial_j - Y^j \partial_j X^i \partial_i) f\end{aligned}$$

Here we have used the commutativity of partial derivatives. This computation gives a formula for the commutator in local coordinates.

Proof of Theorem 7. Compare with [Roe]. Letting $g_{jk} = (\partial_j, \partial_k)$, the compatibility of ∇ with g gives

$$\partial_i g_{jk} = \sum_a \Gamma^a_{ij} g_{ak} + \Gamma^a_{ik} g_{aj} \tag{2}$$

and by permuting (i, j, k) we get

$$\partial_j g_{ki} = \sum_a \Gamma^a_{jk} g_{ai} + \Gamma^a_{ji} g_{ak} \tag{3}$$

$$\partial_k g_{ij} = \sum_a \Gamma^a_{ki} g_{aj} + \Gamma^a_{kj} g_{ai} \tag{4}$$

The symmetry of ∇ simply means $\Gamma^a_{ij} = \Gamma^a_{ji}$. By calculating (2)+(3)-(4) we get

$$\sum_a g_{ak} \Gamma^a_{ij} = \frac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right)$$

Since at every point g_{ak} is an invertible matrix, we see that Γ^a_{ij} is uniquely determined by g. Conversely, this equation may used as a definition which has the required properties. ☐

1.9 Curvature

Definition 8. *Let ∇ be a connection on $V \to M$. For $X, Y \in C^\infty(TM)$ and $Z \in C^\infty(V)$ we define the* curvature *transformation by*

$$K(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z.$$

It may be viewed as a map

$$C^\infty(TM) \otimes C^\infty(TM) \otimes C^\infty(V) \to C^\infty(V).$$

Note that $\nabla_{[X,Y]} Z = 0$ for coordinate vector fields $X = \partial_i, Y = \partial_j$.

Proposition 9. *K is a tensorial in all of its three arguments. That is,*

$$K(fX, Y)Z = K(X, fY)Z = K(X, Y)fZ = fK(X, Y)Z.$$

Proof. For example, using $[fX, Y] = f[X, Y] - Y(f)X$ we have

$$\begin{aligned} K(fX, Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z \\ &= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - \nabla_Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + \nabla_Y(f)\nabla_X Z \\ &= fK(X, Y)Z. \end{aligned}$$ □

Moreover, $K(X, Y)$ is antisymmetric in X, Y. Hence we may view K as an element $K \in \Omega^2(\mathrm{End}(V))$.

Remark 10. *For a finite-dimensional vector space V, the exterior power $\Lambda^m V^*$ may be identified with the space of antisymmetric maps $V^{\otimes m} \to \mathbb{R}$. This identification is given by*

$$\varphi_1 \wedge \ldots \wedge \varphi_m \mapsto \sum_\sigma \mathrm{sgn}(\sigma)\varphi_{\sigma(1)} \otimes \ldots \otimes \varphi_{\sigma(m)},$$

where we sum over the symmetric group on m letters. Here, $\varphi_{\sigma(1)} \otimes \ldots \otimes \varphi_{\sigma(m)}$ is the map that takes $v_1 \otimes \cdots \otimes v_m$ to $\varphi_{\sigma(1)}(v_1)\cdots\varphi_{\sigma(m)}(v_m)$. Note also that some authors use a different convention, where a factor $1/m!$ is introduced.

We return now to a Riemannian manifold (M, g) with its Levi-Civita connection. We then write $R = K$. Let $(e_1, \ldots, e_n)$ be local frame for (TM, g). Then we may introduce functions (R^i_{ljk}) by the requirement

$$R(e_j, e_k)e_l = \sum R^i_{ljk} e_i.$$

For the Levi-Civita connection, the transformation R has a number of symmetries (whose verification is a tedious calculation):

$$\begin{aligned} R(X, Y)Z + R(Y, X)Z &= 0 \\ R(X, Y)Z + R(Y, Z)X + R(Z, X)Y &= 0 \quad \textit{First Bianchi Identity} \\ (R(X, Y)Z, W) + (R(X, Y)W, Z) &= 0 \\ (R(X, Y)Z, W) - (R(Z, W)X, Y) &= 0 \end{aligned} \tag{5}$$

We may also view R as a 4-tensor $g(R(X, Y)Z, W)$. It has the components

$$R_{iljk} = g_{is}R^s_{ljk} = g(R(e_j, e_k)e_l, e_i) = g(R(e_i, e_l)e_k, e_j).$$

1.10 Ricci and scalar curvature

Using the curvature tensor we may form the *Ricci curvature*

$$\mathrm{Ric}(Y, Z) = \mathrm{tr}(X \mapsto R(X, Y)Z),$$

which is symmetric in Y, Z by the last equation in (5). The components of Ric are

$$\mathrm{Ric}_{ab} = \mathrm{Ric}(e_a, e_b) = R^i_{aib}.$$

This $(2, 0)$-tensor may also be viewed as an endomorphism $\mathrm{Rc}\colon TM \to TM$, determined by the formula (here we use the non-degeneracy of the metric g)

$$\mathrm{Ric}(X, Y) = g(X, \mathrm{Rc}(Y)).$$

In components, $\mathrm{Rc}^b_a = g^{bj}\mathrm{Ric}_{ja}$, where (g^{ij}) denotes the inverse of the matrix (g_{ij}). The *scalar curvature* $\kappa\colon M \to \mathbb{R}$ is defined by taking the trace again:

$$\kappa = \mathrm{scal}_g = \mathrm{tr}(\mathrm{Rc}) = \mathrm{Rc}^a_a = g^{ab}\mathrm{Ric}_{ab}$$

Sometimes we write scal_g instead of κ (which is preferred by Roe).

1.11 Normal coordinates

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇.

Definition 11. *A curve $\gamma\colon [a,b] \to M$ is called a* geodesic *if the acceleration $\nabla_t \dot\gamma = 0$ vanishes, where the velocity $\dot\gamma$ is viewed as a vector field along γ.*

Using the compatibility of ∇ with g we see that

$$\frac{d}{dt} g(\dot\gamma, \dot\gamma) = 2g(\nabla_t \dot\gamma, \dot\gamma).$$

Hence the velocity of a geodesic $\|\dot\gamma\|$ is constant. In local coordinates, $\gamma = (\gamma^1, \ldots, \gamma^n)$ and the defining condition for a geodesic becomes

$$0 = \nabla_t \dot\gamma = \ddot\gamma^j \partial_j + \dot\gamma^j \nabla_{\dot\gamma^i \partial_i} \partial_j = (\ddot\gamma^k + \dot\gamma^i \dot\gamma^j \Gamma^k_{ij}) \partial_k$$

This leads to the *geodesic equations*

$$\ddot\gamma^k + \dot\gamma^i \dot\gamma^j \Gamma^k_{ij} = 0 \qquad (k = 1, \ldots, n).$$

From the theory of ODEs of second order we obtain:

Theorem 12. *Through every point $p \in M$ there exists a unique*[1] *geodesic $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, with given velocity vector $\dot\gamma(0) = v \in T_pM$.*

Using geodesics we may form the *exponential map* in Riemannian geometry:

$$\exp\colon U \to M, \qquad v \mapsto \gamma_v(1)$$

is defined in some neighborhood $U \subset T_pM$ of 0. Here γ_v denotes the unique geodesic with $\gamma(0) = p, \dot\gamma(0) = v$. This defines a smooth map and the differential of exp at 0 is the identity map (see the exercise sheet)

$$T_0(T_pM) \cong T_pM \to T_pM.$$

By the Inverse Function Theorem, shrinking if necessary the neighborhood U, we obtain a diffeomorphism $\exp\colon U \to \exp(U)$ onto its open image $\exp(U) \subset M$.

By choosing of an orthonormal basis e_k in T_pM we may view U as an open subset of $\mathbb{R}^n$. Thus $\exp^{-1}$ gives us a local chart on $\exp(U) \subset M$ and correspondingly *normal coordinates* $(x^1, \ldots, x^n)$ based at p on M. The correspondence is given by

$$U \ni (x^1, \ldots, x^n) \mapsto \exp(x^1 e_1 + \cdots + x_n e_n) \in M.$$

Clearly, we have $g_{ij}(p) = \delta_{ij}$ for normal coordinates based at p.

Proposition 13. *In normal coordinates based at p, all the Christoffel symbols $\Gamma^k_{ij}(p) = 0$ vanish at p.*

Proof. We first observe that $\gamma_v : t \mapsto \exp(tv)$ for $v \in \mathbb{R}^n$ represents a geodesic through p in normal coordinates based at p, so that $\nabla_t \dot\gamma(0) = 0$. This follows because $\gamma_{tv}(s) = \gamma_v(ts)$ on behalf of the uniqueness of geodesics. We then get

$$0 = \nabla_{\partial_i + \partial_j}(\partial_i + \partial_j) = \nabla_i \partial_i + \nabla_j \partial_j + 2\nabla_i \partial_j = 2\nabla_i \partial_j$$

□

Remark 14. *1. On a Riemannian manifold, we may introduce the* distance

$$d(p,q) = \inf\{\text{length}(\gamma) \mid \gamma\colon [0,1] \to M, \gamma(0) = p, \gamma(1) = q\}.$$

This defines a metric d on M.

[1] More precisely, a unique germ of a geodesic

2. *For any $p \in M$ we find $\varepsilon > 0$ so that any q with $d(p,q) < \varepsilon$ may be joined to p by a unique constant speed geodesic γ of constant velocity $\|\dot{\gamma}\|$ (here we have used that ∇ is symmetric).*

3. *In normal coordinates, the ball of radius ε in $U \subset T_pM$ corresponds to the d-ball of radius ε in M.*

4. *The* Theorem of Hopf-Rinow *asserts that (M,d) is a complete metric space precisely when any two points p,q on M may be joined by a geodesic* (geodesic completeness).

1.12 Families of geodesics and Jacobi fields

Consider a smooth map $\varphi\colon \mathbb{R}^2 \supset U \to M$. We will use the notation

$$\partial_s\varphi = \frac{\partial\varphi}{\partial s}(s,t), \qquad \partial_t\varphi = \frac{\partial\varphi}{\partial t}(s,t).$$

This defines two vector fields along φ. For instance, $\partial_s\varphi(s,t) \in T_{\varphi(s,t)}M$ is the velocity at s of the curve $\varphi(s,t)$, where t is being held fixed. Since ∇ is symmetric, we have (see exercise sheet 2)

$$\nabla_t\partial_s\varphi = \nabla_s\partial_t\varphi. \tag{6}$$

Suppose all $\varphi(s,-)$ are geodiscs. The *variational vector field* or the *Jacobi field* is then defined by

$$J(t) = \frac{\partial}{\partial s}\bigg|_{s=0}\varphi(-,t) = \partial_s\varphi(0,t)$$

It is a vector field along $\gamma(t) = \varphi(0,t)$. Using (6) and the definition of R we find

$$\nabla_t\nabla_t\partial_s\varphi = \nabla_t\nabla_s\partial_t\varphi = \nabla_s\nabla_t\partial_t\varphi + R(\partial_t\varphi,\partial_s\varphi)\partial_t\varphi$$

By evaluating this expression at $s=0$ one gets the following:

Proposition 15. *Suppose every $\varphi(s,-)$ is a geodesic. Then we have the* Jacobi equation

$$\nabla_t^2 J = R(\dot{\gamma},J)\dot{\gamma}.$$

1.13 Geometric interpretation of curvature via Taylor expansion of the metric

Proposition 16. *Let $(x^1,\dots,x^n)$ be normal coordinates based at $p \in M$. Near p we then have an expansion*

$$g_{ij} = \delta_{ij} + \frac{1}{3}R_{kijl}x^kx^l + O(\|x\|^3), \tag{7}$$

where R_{kijl} is the Riemannian curvature tensor at the point p.

Proof. We perform all calculations on $U \subset T_pM = \mathbb{R}^n$ on which the exponential map is a diffeomorphism. In particular, g denotes the Riemannian metric on U induced by the given Riemannian metric g on M and the exponential map and ∇ denotes the induced covariant derivative on TU.

For $\alpha \in T_0(U) = T_pM$ with $\|\alpha\| = 1$ we have a geodesic $\gamma_\alpha : t \mapsto t\alpha$ of unit speed in (U,g).

Using a vector $\beta \in T_0(U)$ we may form the family of geodesics

$$\varphi(s,t) = (\alpha + s\beta)t.$$

The Jacobi field is given by $J(t) = t\beta$ along γ_α (we identify $T_{t\alpha}(U) = T_pM$ for all t). Let $f(t) = g_{t\alpha}(t\beta,t\beta)$, where $g_{t\alpha}$ denotes the metric g at the point $t\alpha \in U \subset T_pM$.

We calculate

$$
\begin{aligned}
f(0) &= 0 \\
f'(t) &= \partial_t g(J,J) = 2g(\nabla_t J, J) \\
f'(0) &= 0 \\
f''(t) &= 2g(\nabla_t^2 J, J) + 2g(\nabla_t J, \nabla_t J) \\
f''(0) &= 2g_0(\beta,\beta) \\
f'''(t) &= 2g(\nabla_t^3 J, J) + 6g(\nabla_t^2 J, \nabla_t J) \\
f'''(0) &= 6g(R(\alpha, J)\alpha, \nabla_t J)|_{t=0} = 0 \\
f''''(t) &= 2g(\nabla_t^4 J, J) + 8g(\nabla_t^3 J, \nabla_t J) + 6g(\nabla_t^2 J, \nabla_t^2 J) \\
f''''(0) &= 8g_0(R(\alpha,\beta)\alpha,\beta).
\end{aligned}
$$

In the fifth line we use that by definition $\nabla_t J(0) = \beta$, because all Christoffel symbols vanish at $0 \in U$ for the normal coordinates $(x^1, \ldots, x^n)$. In the last line we use the fact that

$$
\nabla_t(\nabla_t^2 J)|_{t=0} = \nabla_t(R(\dot\gamma_\alpha, J(t))\dot\gamma_\alpha)|_{t=0} = \nabla_t(R(\alpha, J)\alpha)|_{t=0} = R(\alpha, \nabla_t J|_{t=0})\alpha = R(\alpha,\beta)\alpha
$$

where in the first equation we use the Jacobi equation and in the third equation we use the following calculation: Consider $t \mapsto R(\alpha, -)\alpha$ as an endomorphism λ along γ_α. In our local coordinates it is given by a time dependent matrix $\lambda_i^j(t)$ so that $\lambda(t)(\partial_i) = \lambda_i^j(t)\partial_j$. Let $J(t) = J^i(t)\partial_i = t\beta^i\partial_i$, where we've set $\beta = \beta^i\partial_i$. Then

$$
\nabla_t(J^i(t)\lambda_i^j(t)\partial_j) = ((J^i)'(t)\lambda_i^j(t) + J^i(t)(\lambda_i^j)'(t))\partial_j + J^i(t)\lambda_i^j(t)\nabla_t\partial_j.
$$

Evaluation at $t = 0$ and using that $J(0) = 0$ (hence $J^i(0) = 0$ for all i) then gives

$$
\nabla_t(J^i(t)\lambda_i^j(t)\partial_j)|_{t=0} = (J^i)'(0)\lambda_i^j(0)\partial_j = \beta^i\lambda(0)(\partial_i) = \lambda(0)(\beta^i\partial_i) = \lambda(0)(\beta)
$$

which is our assertion.

Altogether we obtain the expansion

$$
g(\beta,\beta)(t\alpha) = \frac{f(t)}{t^2} = g_0(\beta,\beta) + \frac{t^2}{3} g_0(R(\alpha,\beta)\alpha,\beta) + O(t^3)
$$

for $t \to 0$. We have

$$
g_0(\beta,\beta) = \delta_{ij}\beta^i\beta^j.
$$

In order to establish equality (7) at a point $q = (x^i) \in U$ of small norm, we take the vector $\alpha = \frac{1}{t}x^i\partial_i$ with t chosen so that α has unit norm. Then

$$
\begin{aligned}
\frac{t^2}{3} g_0(R(\alpha,\beta)\alpha,\beta) &= \frac{1}{3} g_0(R(x^k\partial_k, \beta^i\partial_i)x^l\partial_l, \beta^j\partial_j) \\
&= \frac{1}{3} R_{kijl} x^k x^l \beta^i \beta^j.
\end{aligned}
$$

Here R_{kijl} are the components of the curvature operator of g at p in the coordinates $(x^1, \ldots, x^n)$. We hence get the expansion

$$
g_{ij}(q)\beta^i\beta^j = \delta_{ij}\beta^i\beta^j + \frac{1}{3} R_{kijl}(p) x^k x^l \beta^i\beta^j + O(\|x\|^3)
$$

for $x \to 0$ and for all vectors $\beta = \beta^i\partial_i \in T_pM$. From polarization we deduce

$$
g_{ij} = \delta_{ij} + \frac{1}{3} R_{kijl} x^k x^l + O(\|x\|^3)
$$

as required. $\square$

We will use this expansion in order to obtain an expansion of the volume density $\sqrt{\det(g)}$ around p. First, using the formula

$$\det(\exp(A)) = \exp(\operatorname{tr}(A))$$

for any $A \in \mathbb{R}^{n\times n}$ and writing

$$g = \exp(C) + O(\|x\|^3)$$

where $C_{ij} = \frac{1}{3}R_{kijl}x^k x^l$ we obtain

$$\det(g) = \exp(\operatorname{tr}(C))+O(\|x\|^3) = 1+\operatorname{tr}(C)+O(\|x\|^3) = 1+\frac{1}{3}\delta^{ij}R_{kijl}x^k x^l+O(\|x\|^3) = 1-\frac{1}{3}\mathrm{Ric}_{kl}x^k x^l+O(\|x\|^3).$$

Here Ric_{kl} are the components of the Ricci tensor at p in normal coordinates $(x^1, \dots, x^n)$. The Taylor expansion

$$\sqrt{y} = 1 + \frac{1}{2}(y-1) + O(|y-1|^2)$$

for $y \to 1$ now leads to the expansion

$$\sqrt{\det(g)} = 1 - \frac{1}{6}\mathrm{Ric}_{kl}x^k x^l + O(\|x\|^3)$$

of the volume densitiy around p. This gives a very nice geometric interpretation of the Ricci tensor: Up to a multiple it can be identified with the Hessian of the volume density around p in normal coordinates.

Theorem 17. *Let ω_n denote the volume of the unit ball in Euclidean space $\mathbb{R}^n$. Let (M, g) be a Riemannian manifold and let $p \in M$. For the volume of the ball $B_r(p)$ of radius r around p (with respect to the Riemannian metric g) we have the expansion*

$$\mathrm{vol}(B_r(p)) = \omega_n r^n \left(1 - \frac{\mathrm{scal}_g(p)}{6(n+2)}r^2 + O(r^4)\right)$$

for $r \to 0$.

Hence the scalar curvature measures the asymptotic volume growth of small balls around p.

Proof. We have

$$\mathrm{vol}(B_r(p)) = \int_{B_r(0)\subset(T_pM,g_p)} \sqrt{\det(g)}d\mathrm{vol} = \int_{B_r(0)} (1 - \frac{1}{6}\mathrm{Ric}_{kl}x^k x^l + O(\|x\|^3))d\mathrm{vol}$$

Note that $B_r(0)$ is the ball of radius r measured with respect to the metric g_p on T_pM, because this is sent to the ball $B_r(p)$ under the exponential map. After choice of an orthonormal basis with respect to g_p we identify T_pM with $\mathbb{R}^n$ equipped with the standard Euclidean scalar product. In the above formula, $d$vol denotes the standard measure on $T_pM = \mathbb{R}^n$ with respect to the Euclidean metric.

Let $d\sigma$ denote the standard volume element of the unit sphere S^{n-1} in $\mathbb{R}^n$ with respect to the Euclidean metric. First, for $k \neq l$ we get

$$\int_{S^{n-1}} x^k x^l d\sigma = 0$$

by the change of variables formula for $x^k \mapsto -x^k$. Recall that the superscripts k and l are just indices, not exponents. The same argument shows that the expansion appearing in Theorem 17 contains only even powers of r.

For $1 \leq i, j \leq n$ we get

$$\int_{S^{n-1}} (x^i)^2 d\sigma = \int_{S^{n-1}} (x^j)^2 d\sigma$$

so that

$$\int_{S^{n-1}} (x^i)^2 d\sigma = \frac{1}{n}\int_{S^{n-1}} d\sigma = \omega_n.$$

Summarizing we have

$$\begin{aligned}\int_{B_r(0)} x^k x^l d\mathrm{vol} &= \delta_{kl} \cdot \int_0^r \int_{S^{n-1}} (\tau x)^k (\tau x)^l \tau^{n-1} d\sigma d\tau \\ &= \delta_{kl} \cdot \frac{r^{n+2}}{n+2} \int_{S^{n-1}} x^k x^l d\sigma \\ &= \delta_{kl} \cdot \frac{\omega_n}{n+2} r^{n+2}.\end{aligned}$$

This yields the formula

$$\int_{B_r(0)} \mathrm{Ric}_{kl} \cdot x^k x^l d\mathrm{vol} = \omega_n \frac{\mathrm{scal(p)}}{n+2} r^{n+2}$$

implying the claim of Theorem 17 . □

2 Dirac bundles and Dirac operators

2.1 The Clifford Algebra

As a motivating example assume that S is a (real or complex) vector space together with a collection of linear maps $J_1, \dots, J_n : S \to S$ satisfying the following identities:

- $J_i^2 = -\mathrm{Id}_S$ for all i,
- $J_i J_j = -J_j J_i$ for alle $i \neq j$.

In this case we define the *Dirac operator* $D : C^\infty(\mathbb{R}^n, S) \to C^\infty(\mathbb{R}^n, S)$ (where $C^\infty(\mathbb{R}^n, S)$ is the space of smooth functions $\mathbb{R}^n \to S$) by

$$D(f) := \sum_{i=1}^n J_i \circ \frac{\partial}{\partial x_i}.$$

This is a linear differential operator whose square is equal to the Laplace operator:

$$D^2 = \Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

It was found by Paul Dirac (1928) in his description of the quantum mechanical behavior of fermions. Lorentz invariance forces the relevant differential operator to be of first order.

Spin geometry arises out of the attempt to replace $\mathbb{R}^n$ by an arbitrary Riemannian manifold (M, g) and hence to define the Dirac operator in a coordinate independent way.

Definition 18. *Let $(V, (-,-))$ be a Euclidean vector space of dimension n. The* Clifford algebra $\mathrm{Cl}(V)$ *is defined as the quotient of the free tensor algebra*

$$\mathrm{Cl}(V) = \bigoplus_{r=0}^\infty V^{\otimes r} \Big/ I$$

modulo the ideal I generated by all elements of the form $v \otimes v + \|v\|^2$ for $v \in V$.

This is an algebra with unit element $1 \in \mathbb{R} = \bigotimes^0 V$ and it contains $V = \bigotimes^1 V$ as a linear subspace. Elements in the Clifford algebra are finite linear combinations of monomials $v_1 v_2 \dots v_r$ $(r \in \mathbb{N})$ subject to the relations

- $v_i v_j = -v_j v_i$ if $v_i \perp v_j$,
- $v_i^2 = -\|v_i\|^2$.

Example 19. *There are isomorphisms of* $\mathbb{R}$*-algebras* $\mathrm{Cl}(1) \cong \mathbb{C}$ *and* $\mathrm{Cl}(2) \cong \mathbb{H}$.

2.2 Relation to the Exterior Algebra

Recall that an element of the exterior algebra $\Lambda^* V$ may be written as a sum of elements

$$v_1 \wedge \ldots \wedge v_r \qquad v_i \in V, r \in \mathbb{N}.$$

More formally, $\Lambda^* V = TV/J$ is the quotient of the tensor algebra $TV = \bigoplus_{r \geq 0} V^{\otimes r}$ modulo the ideal J generated by elements of the form $v \otimes v$ for $v \in V$. We will compare the vector space $\mathrm{Cl}(V)$ with $\Lambda^*(V)$ via the map

$$\lambda \colon \Lambda^k V \to \mathrm{Cl}(V), \quad v_1 \wedge \ldots \wedge v_k \mapsto \frac{1}{k!} \sum_{\sigma} \operatorname{sgn}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)}. \tag{8}$$

Proposition 20. *The maps* (8) *define an isomorphism* $\lambda \colon \Lambda^*(V) \to \mathrm{Cl}(V)$ *of vector spaces* (not *of algebras*).

Proof. Consider the canonical projection

$$\pi \colon \bigoplus_{r \geq 0} V^{\otimes r} \to \mathrm{Cl}(V)$$

and let $\mathrm{Cl}^{(k)}(V) = \pi \left(\bigoplus_{r=0}^{k} V^{\otimes r} \right) \subset \mathrm{Cl}(V)$. For each k we have an induced map

$$\Lambda^k(V) \to \mathrm{Cl}^{(k)}(V) / \mathrm{Cl}^{(k-1)}(V), \quad v_1 \wedge \ldots \wedge v_k \mapsto v_1 \cdots v_k$$

which is induced by (8) and clearly surjective. Hence the map $\Lambda^*(V) \to \mathrm{Cl}(V)$ is also surjective.

We shall prove that $\dim \mathrm{Cl}(V) = 2^n$, which will then complete the proof, as $\dim \Lambda^*(V) = 2^n$ as well. Recall that a $\mathbb{Z}/2$-*graded algebra* is an algebra C with a decomposition $C = C^0 \oplus C^1$ with $C^i C^j \subset C^{i+j}$, taking indices mod two. It is called *graded commutative* if $xy = (-1)^{ij} yx$ whenever $x \in C^i, y \in C^j$. Given two graded commutative algebras $C = C^0 \oplus C^1$, $D = D^0 \oplus D^1$, their *graded tensor product* is again graded by

$$(C \tilde{\otimes} D)^0 = (C^0 \otimes D^0) \oplus (C^1 \otimes D^1)$$
$$(C \tilde{\otimes} D)^1 = (C^1 \otimes D^0) \oplus (C^0 \otimes D^1)$$

The multiplication on $C \tilde{\otimes} D$ is given by

$$(c_1 \otimes d_1)(c_2 \otimes d_2) = (-1)^{|d_1||c_2|} (c_1 c_2) \otimes (d_1 d_2)$$

and is also graded commutative.

An important example is the Clifford algebra, which is graded commutative, where we set

$$\mathrm{Cl}^0(V) = \pi \left(\bigoplus_{r \geq 0} V^{\otimes 2r} \right), \quad \mathrm{Cl}^1(V) = \pi \left(\bigoplus_{r \geq 0} V^{\otimes 2r+1} \right).$$

Alternatively, consider $\alpha \colon V \to V, \alpha(v) = -v$. Then α induces a map $\mathrm{Cl}(V) \to \mathrm{Cl}(V)$ since $\alpha(v) \cdot \alpha(v) = v \cdot v = -\|v\|^2$. Then $\mathrm{Cl}^0(V)$ is the $(+1)$-eigenspace and $\mathrm{Cl}^1(V)$ is the (-1)-eigenspace of this map. Sometimes we call $\mathrm{Cl}^0(V)$ the *even* part of $\mathrm{Cl}(V)$ and $\mathrm{Cl}^1(V)$ the *odd part* of $\mathrm{Cl}(V)$. Note that the even part forms a subalgebra of $\mathrm{Cl}(V)$, but not the odd part.

Now if V is the orthogonal direct sum of V_1 and V_2, then we have

$$\mathrm{Cl}(V) \cong \mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2).$$

For the proof, we define two homomorphisms inverse to each other. First,

$$V = V_1 \oplus V_2 \to \mathrm{Cl}(V_1) \tilde{\otimes} \mathrm{Cl}(V_2), \quad (v_1, v_2) \mapsto v_1 \otimes 1 + 1 \otimes v_2$$

induces an algebra map $f\colon \mathrm{Cl}(V) \to \mathrm{Cl}(V_1)\tilde{\otimes}\,\mathrm{Cl}(V_2)$ because $(v_1 \otimes 1 + 1 \otimes v_2)^2 = -\|v_1\|^2 - \|v_2\|^2 = -\|(v_1, v_2)\|^2$.

The map $g\colon \mathrm{Cl}(V_1)\tilde{\otimes}\,\mathrm{Cl}(V_2) \to \mathrm{Cl}(V)$ is induced by the bilinear map

$$\mathrm{Cl}(V_1) \times \mathrm{Cl}(V_2) \to \mathrm{Cl}(V), \quad (x, y) \mapsto x \cdot y$$

where we view $x \in \mathrm{Cl}(V)$ by the canonical map $\mathrm{Cl}(V_1) \to \mathrm{Cl}(V)$ induced by the inclusion $V_1 \hookrightarrow V \to \mathrm{Cl}(V)$ and similarly for y.

It is a straightforward calculation that f and g are indeed inverse to each other.

Since $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}$ and since we may orthogonally decompose $V = V_1 \oplus \cdots \oplus V_n$ into 1-dimensional subspaces, it follows that

$$\mathrm{Cl}(V) = \mathrm{Cl}(V_1)\tilde{\otimes} \cdots \tilde{\otimes}\,\mathrm{Cl}(V_n), \tag{9}$$

which has dimension 2^n. $\square$

From (9) we also see the following:

Proposition 21. *Let $(v_1, \ldots, v_n)$ be a basis of V. Then the elements*

$$v_{i_1} \cdots v_{i_k} \qquad i_1 < \cdots < i_k, \;\; 0 \le k \le \dim V$$

form a basis of $\mathrm{Cl}(V)$. Hence $\dim_{\mathbb{R}} \mathrm{Cl}(V) = 2^n$.

For $v \in V$ we define the *contraction* by v as

$$\iota_v\colon \Lambda^k V \to \Lambda^{k-1} V, \quad v_1 \wedge \cdots \wedge v_k \mapsto \sum_{i=1}^{k} (-1)^{i+1} (v, v_i) v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k.$$

Roe considers the closely related operator $v \lrcorner - := -\iota_v$. The contraction has the following properties:

1. $\iota_v \iota_v \omega = 0$. It follows from the universal property that ι induces a map $\Lambda^*(V) \times \Lambda^*(V) \to \Lambda^*(V)$.
2. Under the canonical isomorphism $\Lambda(V) \cong \Lambda(V^*)$ (given by taking the k-th exterior power of the map $V \cong V^*, v \mapsto (v, -)$) the contraction corresponds to the map

$$\begin{aligned} \iota_v\colon \Lambda^{k+1}(V^*) &\to \Lambda^k(V^*), \\ (\iota_v \omega)(X_1, \ldots, X_k) &= \omega(v, X_1, \ldots, X_k) \end{aligned}$$

   Again, notice that working with Roe's operator we have $(v \lrcorner \omega) = -\omega(v, X_1, \ldots, X_k)$

Proposition 22. *Under the isomorphism $\lambda\colon \Lambda^* V \to \mathrm{Cl}(V)$ of vector spaces defined above, the multiplication on $\mathrm{Cl}(V)$ corresponds to the product*

$$\Lambda^*(V) \times \Lambda^*(V) \to \Lambda^*(V), \quad (v, \omega) \mapsto v \wedge \omega - \iota_v \omega = v \wedge \omega + v \lrcorner \omega.$$

Proof. Fix $v \in V$ and extend $v = v_1$ to an orthonormal basis $(v, v_2, \ldots, v_n)$. Suppose $w = v_{i_1} \wedge \cdots \wedge v_{i_k}$. Then in the Clifford algebra we have

$$\lambda(v)\lambda(w) = v v_{i_1} \cdots v_{i_k} = \begin{cases} v_1 v_{i_1} \cdots v_{i_k} = \lambda(v \wedge w) & i_1 > 1 \\ -\lambda(v_{i_2} \wedge \cdots \wedge v_{i_k}) = -\lambda(\iota_v w) & i_1 = 1. \end{cases}$$

This equals $\lambda(v \wedge w - \iota_v w)$ in both cases. $\square$

2.3 Hodge Star Operator

Let $(v_1, \dots, v_n)$ be a positively ordered orthonormal basis of V. The *volume element* is defined by

$$\mathrm{vol} = v_1 \wedge \cdots \wedge v_n \in \Lambda^n(V).$$

The inner product of V is extended to $\Lambda^k V$ by declaring $v_{i_1} \wedge \ldots \wedge v_{i_k}$ to be an orthonormal basis of $\Lambda^k V$.

Definition 23. *The* Hodge star *operator* $*\colon \Lambda^k V \to \Lambda^{n-k} V$ *is defined by the relation*

$$(\alpha, \beta)\mathrm{vol} = \beta \wedge *\alpha \qquad \forall \beta \in \Lambda^k V.$$

In particular, $*(v_1 \wedge \cdots \wedge v_k) = v_{k+1} \wedge \cdots \wedge v_n$. A useful formula is

$$*(v_{i_1} \wedge \cdots \wedge v_{i_k}) = \pm v_{j_1} \wedge \cdots \wedge v_{j_{n-k}}$$

where the j's are chosen so that $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ is a permutation of $(1, \dots, n)$ and the sign is chosen according to the parity of this permutation.

Lemma 24. $v \lrcorner \omega = (-1)^{nk+n+1} * (v \wedge *\omega)$ *for* $\omega \in \Lambda^k V$. *In this sense the contraction is dual to the wedge product.*

2.4 Clifford Modules

Definition 25. *A real (resp. complex)* Clifford module *for* $\mathrm{Cl}(V)$ *is a real (resp. complex) vector space* S *together with an* $\mathbb{R}$*-algebra map*

$$\rho : \mathrm{Cl}(V) \to \mathrm{Hom}(S, S).$$

(*more precisely,* $\mathrm{Hom}_{\mathbb{R}}(S, S)$ *resp.* $\mathrm{Hom}_{\mathbb{C}}(S, S)$) *Equivalently this is given by an* $\mathbb{R}$*-linear map* $c : V \to \mathrm{Hom}(S, S)$ *such that* $c(v)^2 = -\|v\|^2 \cdot \mathrm{id}_S$.

Example 26. *In the situation at the beginning of this section we can regard* S *as a Clifford module for* $\mathrm{Cl}(n)$ *by setting* $e_i \mapsto J_i$ *for the standard orthonormal basis* $(e_1, \dots, e_n)$ *of* $\mathbb{R}^n$ *and extending this map to an algebra map* $\mathrm{Cl}(\mathbb{R}^n) \to \mathrm{Hom}(S, S)$. *More generally, for a finite-dimensional Euclidean vector space* V *and any choice of orthonormal basis* $(v_1, \dots, v_n)$ *of* V *the Dirac operator on* $C^\infty(V, S)$ *is given by*

$$D(f) = \sum_{i=1}^{n} v_i \cdot \partial_i f,$$

where, as usual, $\partial_i = \frac{\partial}{\partial v^i}$ *is the derivative in the direction* v_i.

This is hence a description of the Dirac operator which is independent of the choice of an orthonormal basis of V*: suppose* $w_j = g^{ji} v_i$ *for* $(g^{ji}) \in O(n)$. *Then*

$$\sum w_j \frac{\partial}{\partial w_j} = \sum_{i,j} g^{ji} v_i g^{jk} \frac{\partial}{\partial v_k} = \sum v_i \frac{\partial}{\partial v_i}$$

since $\sum_j g^{ji} g^{jk} = \delta^{ik}$.

In this lectures, we will usually restrict our attention to complex Clifford modules S.

Example 27. $S = \mathrm{Cl}(V)$ *is itself a Clifford module, where the module structure is given by multiplication in the Clifford algebra, i.e. by* $\mathrm{Cl}(S) \times S \to S, (v, w) \mapsto v \cdot w$.

Now let (M,g) be a Riemannian manifold. Because each fibre T_pM of the tangent bundle is a Euclidean vector space, we can form its Clifford algebra $\mathrm{Cl}(T_pM)$. This construction can be carried out on local trivializations of TM and hence yields a vector bundle $\mathrm{Cl}(TM) \to M$ of dimension 2^n. This bundle is a bundle of algebras in the sense that we have a smooth map

$$\mathrm{Cl}(TM) \oplus \mathrm{Cl}(TM) \to \mathrm{Cl}(TM)$$

which restricts to Clifford multiplication in each fibre. We call the bundle $\mathrm{Cl}(TM) \to M$ the *bundle of Clifford algebras* associated to $TM \to M$.

In the following we use the notion *Hermitian bundle* for a complex vector bundle $V \to M$ which is equipped with a fiber wise Hermitian inner product $(-,-)$. We also could call these bundles *complex bundles with inner product*. A *compatible connection* is a connection ∇ on V which is compatible with the Hermitian structure in the sense of Definition 5, respectively Proposition 6.

Definition 28. *Let M be a Riemannian manifold. A* bundle of Clifford modules *or* Clifford-module bundle *or briefly* Clifford bundle *for $\mathrm{Cl}(TM)$ is a Hermitian vector bundle $S \to M$ together with a smooth bundle map*

$$\mathrm{Cl}(TM) \oplus S \to S \tag{10}$$

with the following properties:

1. *The map (10) restricts to a Clifford module structure $\mathrm{Cl}(T_pM) \times S_p \to S_p$ on each fiber over $p \in M$.*

2. *Clifford multiplication is compatible with the Hermitian structure on S: If $v \in C^\infty(TM)$ is a (local) section of constant length 1, then*

$$(vs_1, vs_2) = (s_1, s_2) \qquad \forall s_1, s_2 \in C^\infty(S).$$

   Note that this is equivalent (using the relation $v^2 = -\|v\|^2$ in $\mathrm{Cl}(V)$), that Clifford multiplication with tangent vectors is skew adjoint (this is the condition given by Roe):

$$(vs_1, s_2) + (s_1, vs_2) = 0 \qquad \forall v \in C^\infty(TM), s_1, s_2 \in C^\infty(S).$$

We remark that any complex bundle $S \to M$ with a fiber wise Clifford-module structure can be equipped with a Hermitian inner product so that the second condition above is satisfied, cf. Exercise 4 on Exercise sheet 3. There is an analogous notion of real Clifford-bundles, but this will be less important for us.

Definition 29. *Let M be a Riemannian manifold. A* Dirac bundle *is a Clifford module bundle $S \to M$ together with a connection ∇^S on S which is compatible with the Hermitian structure and with the Clifford module structure in the sense that*

$$\nabla^S_X(vs) = \nabla^{\mathrm{LC}}_X(v) \cdot s + v\nabla^S_X(s)$$

for all $X, v \in C^\infty(TM)$, $s \in C^\infty(S)$ using the Levi-Civita connection ∇^{LC} on TM.

Remark 30. *What we call Dirac bundle is called* Clifford bundle *by Roe. We find the above notions less confusing.*

Definition 31. *Let S be Dirac bundle on a Riemannian manifold M. Then the corresponding* Dirac operator *is the composition*

$$D\colon C^\infty(S) \xrightarrow{\nabla^S} C^\infty(T^*M \otimes S) \cong C^\infty(TM \otimes S) \to C^\infty(S)$$

In terms of a local orthonormal frame (v_i) of TM this may be rewritten as

$$Ds = \sum v_i \cdot \nabla_{v_i} s,$$

similar as before (note that the Hermitian metric plays no role in this definition).

2.5 The Laplacian on Manifolds

Definition 32. *Let $E \to M$ be a vector bundle with connection ∇. The* second covariant derivative *of $s \in C^\infty(E)$ is*

$$\nabla^2_{X,Y}(s) = \nabla_X \nabla_Y(s) - \nabla_{\nabla_X Y}(s).$$

The second term has been inserted to make this a tensor in both X and Y. Given an inner product on TM (i.e. a Riemannian metric on M), the connection Laplacian *is*

$$\Delta(s) = -\sum_{i=1}^{n} \nabla_{e_i,e_i}(s) \tag{11}$$

for a local orthonormal frame (e_i) of TM. It may be regarded as a map $\Delta\colon C^\infty(E) \to C^\infty(E)$.

Note that this is the usual Laplace operator on $C^\infty(\mathbb{R}^n)$ when $M := \mathbb{R}^n$ is equipped with the standard Euclidean structure and smooth functions on M are regarded as sections in the trivial bundle $E = M \times \mathbb{R} \to M$.

Theorem 33 (Weitzenböck Formula)**.** *Let $S \to M$ be a Dirac bundle. Then we have*

$$D^2(s) = \Delta(s) + K(s), \qquad \forall s \in C^\infty(S),$$

where $K \in C^\infty(\mathrm{End}(S))$ is the endomorphism of S given by $K(s) = \sum_{i<j} e_i e_j \cdot K^S(e_i, e_j)(s)$ for the curvature K^S of S.

Note that Δ and D^2 are differential operators of second order, while K is an endomorphism of S (a differential operator of 0th order). The Weitzenböck formula says that the Dirac operator squares to the Laplacian, up to an operator of order 0 (i.e. up to a section in the endomorphism bundle of S), which is related to the curvature of the connection ∇^S on S.

Proof. Choose an orthonormal frame (e_i) of TM around $p \in M$ such that for all i, j we have

$$\nabla_{e_i}(e_j)(p) = 0$$

at p (one says the frame is *synchronous* in p). This can be done by parallel extending an orthonormal basis of T_pM along radial geodesics starting at p. We compute in the point $p \in M$:

$$\begin{aligned}
D^2(s)(p) &= \sum_{i,j} e_j \nabla_{e_j} \left(e_i \nabla_{e_i}(s)\right) \\
&= \sum_{i,j} e_j e_i \nabla_{e_j} \nabla_{e_i}(s) \\
&= -\sum_i \nabla_i^2(s) + \sum_{j<i} e_j e_i (\nabla_{e_j}\nabla_{e_i} - \nabla_{e_i}\nabla_{e_j})(s) = \Delta(s)(p) + K(s)(p)
\end{aligned}$$

where we have used $[e_i, e_j](p) = \nabla_{e_i} e_j - \nabla_{e_j} e_i = 0$ at p. □

2.6 The Bundle of Exterior Forms as a Dirac Bundle

Let $S = \mathrm{Cl}(TM) \otimes \mathbb{C} \to M$. This is a complex Clifford module in an obvious way. Recall that a connection may be defined by specifying which frames are parallel along a given curve γ. Let $(e_1, \dots, e_n)$ be a parallel orthonormal frame of TM along γ. We then define the connection (and the Hermitian metric) on S by declaring the frame

$$(e_{i_1} \cdots e_{i_k})_{1 \le i_1 < \cdots i_k \le n}, \qquad {\scriptstyle 0 \le k \le n},$$

to be a parallel orthonormal frame of S along γ.

We prove $(vs_1, vs_2) = (s_1, s_2)$. For this, extend $e_1 = v$ to an orthonormal frame $(e_1, \ldots, e_n)$. Then by considering cases one immediately checks

$$(ve_{i_1,\ldots,i_k}, ve_{j_1,\ldots,j_k}) = (e_{i_1,\ldots,i_k}, e_{j_1,\ldots,j_k}).$$

The compatibility of the Clifford multiplication with the connection follows from the Leibniz rule (when working in a parallel frame).

We wish to explicitly identify D and K. Recall that $\mathrm{Cl}(V) \cong \Lambda^* V$, where $v \cdot \omega = v \wedge \omega - \iota_v \omega$. Therefore $S \cong \Lambda^* TM \otimes \mathbb{C} \cong \Lambda^*(T^*M) \otimes \mathbb{C}$ identifies with the bundle of differential forms. To identify the Dirac operator D we begin with the following lemma:

Lemma 34. *Let (ε^i) be the frame dual to some orthonormal frame (e_i). For $\omega \in C^\infty(\Lambda^k T^*M)$ we have*

$$d\omega = \sum_{i=1}^n \varepsilon^i \wedge \nabla_{e_i} \omega$$
$$\delta\omega = \sum_{i=1}^n e_i \lrcorner \nabla_{e_i} \omega$$

Proof. It is easy to see that if this formula holds in one orthonormal frame, it holds in any. Let $(x^1, \ldots, x^n)$ be normal coordinates in p. The coordinate vector fields $(\partial_1, \ldots, \partial_n)$ then give an orthonormal basis at p. Using these, we shall prove that the above formula holds in p. Let $\omega = \sum_I \omega_I dx^I$. Then since $\nabla_{\partial_i} \partial_j = 0$ we have at p

$$d\omega(p) = \sum_{i=1}^n \frac{\partial \omega_I}{\partial x_i} dx^i \wedge dx^I = \sum_{i=1}^n dx^i \wedge \nabla_{\partial_i} \omega,$$

as required. Recall that the codifferential of a k-form $\omega \in \Omega^k(M)$ is defined by

$$\delta(\omega) = d^*(\omega) = (-1)^{nk+n+1} * d(*\omega) \in \Omega^{k-1}(M). \tag{12}$$

(the operator d^* may be viewed as the adjoint of d for the L^2 inner product on $\Omega^*(M)$ we will consider later) The second formula now follows from Lemma 24:

$$\begin{aligned} d^*\omega &= (-1)^{nk+n+1} \sum *(e_i \wedge \nabla_{e_i}(*\omega)) = (-1)^{nk+n+1} \sum *(e_i \wedge *\nabla_{e_i}(\omega)) \\ &= \sum e_i \lrcorner \nabla_{e_i} \omega \end{aligned}$$

□

It follows that

$$D\omega = \sum e_i \cdot \nabla_{e_i} \omega = \sum \varepsilon^i \wedge \nabla_{e_i} \omega + \sum e_i \lrcorner \nabla_{e_i} \omega = d\omega + d^*\omega, \tag{13}$$

which is called the *de Rham operator* of M. The corresponding connection Laplacian

$$D^2 = (d + d^*)^2 = dd^* + d^*d$$

is called the *Hodge Laplacian* of M. In this case the Weitzenböck formula $D^2 = \Delta + K$ contains the operator

$$\begin{aligned} K(e_k) &= \sum_{i<j} e_i e_j R(e_i, e_j) e_k \\ &= \frac{1}{2} \sum_{i,j,l} e_i e_j e_l (R(e_i, e_j) e_k, e_l) = \frac{1}{2} \sum_{i,j,l} e_i e_j e_l R_{lkij}. \end{aligned}$$

for the curvature R of the Levi-Civita connection on M. If i, j, l are distinct, by the Bianchi identity we have

$$e_i e_j e_l = e_j e_l e_i = e_l e_i e_j, \qquad R_{lkij} + R_{ikjl} + R_{jkli} = 0.$$

All such summands add up to zero. If $i = j$ we have $R_{lkij} = 0$, if $i \neq l = j$ we have

$$\frac{1}{2} \sum_{i,j} -e_i R_{jkij} = \frac{1}{2} \sum_i e_i \mathrm{Ric}_{ki}.$$

Finally, for $i = l \neq j$ we have

$$\frac{1}{2} \sum e_j R_{ikij} = \frac{1}{2} \sum e_j \mathrm{Ric}_{kj}.$$

Therefore

$$K(e_k) = \sum \mathrm{Ric}_{ak} e_a = \mathrm{Rc}(e_k)$$

is the Ricci transformation. Hence

$$D^2 = \Delta + \mathrm{Rc}.$$

For M closed oriented with $\dim H^1(M;\mathbb{R}) \neq 0$ this may be used to deduce that M does not admit a metric of positive Ricci curvature (meaning that Ric is positive definite at every point). This argument will be made more precise later in these lectures.

3 Spin Structures on Manifolds

3.1 Constructing Clifford Bundles as Associated Bundles

Let M be an oriented Riemannian manifold of dimension n.

In the following we will discuss a general method to construct Clifford and Dirac bundles on M. Let W be some (complex) Clifford representation for $\mathrm{Cl}(n)$, equipped with a compatible (Hermitian) inner product (i.e. vectors of length one act as isometries, or equivalently that Clifford multiplication is skew adjoint). At first we would like to construct a Clifford bundle $E \to M$ whose fibers E_p are isomorphic to W as $\mathrm{Cl}(n)$-modules.

Let $(U_i)_{i \in I}$ be an open cover of M, which is trivializing for TM. Choose orthogonal trivializations $\phi_i : TM|_{U_i} \cong U_i \times \mathbb{R}^n$ with induced transition maps $\phi_j \circ \phi_i^{-1}$ that may be regarded as maps

$$\phi_{ji} \colon U_i \cap U_j \to \mathrm{SO}(n).$$

We wish to define the bundle $E \to M$ of Clifford modules by writing down trivial bundles $U_i \times W$ and choosing suitable transition maps $\psi_{ji} : U_i \cap U_i \to \mathrm{Aut}(W)$. The bundle E is then obtained by gluing the trivial bundles $U_i \times W$ using the transition functions:

$$E = \left(\bigcup_i (U_i \times W) \right) \Big/ \sim \quad \text{where} \qquad ((x, w) \in U_i \times W) \sim ((x, (\psi_{ji})_x(w)) \in U_j \times W)$$

These need to satisfy the following requirements:

- To get a well defined Clifford multiplication on the fibers of the resulting bundle, the transition maps ϕ_{ji} need to be compatible with ψ_{ji} in the following way:

$$(\phi_{ji})_x(v) \cdot (\psi_{ji})_x(w) = (\psi_{ji})_x(vw)$$

  For all $x \in U_i \cap U_j$, all $v \in \mathbb{R}^n$ and $w \in W$.

- They must fulfill the *cocycle condition*

$$\psi_{ki} = \psi_{kj} \circ \psi_{ji} : U_i \cap U_j \cap U_k \to \mathrm{Aut}(W)$$

  for all $i, j, k \in I$ with $U_i \cap U_j \cap U_k \neq \emptyset$.

Each $v \in \mathbb{R}^n$ of norm one induces an isometry of W by $w \mapsto v \cdot w$. We would like to use such isomorphisms to define the maps ψ_{ji}. To encapsulate the geometry of M these should be related to the $SO(n)$-valued transition functions ϕ_{ji}.

3.2 The Pin and Spin Group

Let $(V, (-,-))$ be a Euclidean vector space. We will be mostly interested in the case $V = \mathbb{R}^n$.

Definition 35. *The subgroup* $\mathrm{Pin}(V)$ *of the multiplicative group* $\mathrm{Cl}(V)^\times$ *of invertible elements is generated by all* $v \in V$ *with* $\|v\| = 1$*. An element thus has the form*

$$v_1 \cdots v_k, \qquad \textit{where } \|v_i\| = 1, \ \ k \in \mathbb{N}.$$

The Spin subgroup $\mathrm{Spin}(V) \subset \mathrm{Pin}(V)$ *is the set*

$$\mathrm{Spin}(V) = \{v_1 \cdots v_k \in \mathrm{Pin}(V) \mid k \textit{ even}\} = \{v \in \mathrm{Pin}(V) \mid \alpha(v) = v\} = \mathrm{Pin}(V) \cap \mathrm{Cl}^0(V) \subset \mathrm{Cl}^0(V).$$

Here we recall that $\alpha : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is the algebra automorphism induced by $V \to V$, $v \mapsto -v$. For any $v \in V$ of unit length consider the map

$$\rho_v \colon \mathrm{Cl}(V) \to \mathrm{Cl}(V), \quad x \mapsto -vxv^{-1} = \alpha(v)xv^{-1}.$$

This formula makes sense for any $v \in \mathrm{Cl}(V)^\times$, i.e. we have a map $\mathrm{Cl}(V)^\times \to \mathrm{Aut}(\mathrm{Cl}(V))$.

Lemma 36. *We have* $\rho_v(V) \subset V$*. In fact,* $\rho_v|_V$ *is the reflection across the hyperplane* $v^\perp$ *orthogonal to* $\|v\| = 1$*.*

Proof. If $x = \lambda v$, then $\rho_v(x) = -\lambda v$. On the other hand, if $x \perp v$, then $\rho_v(x) = -vxv^{-1} = xvv^{-1} = x$. This proves that $\rho_v|_V$ is a reflection across $v^\perp$. The first claim follows immediately. □

It follows that ρ_v determines a group homomorphism

$$\rho \colon \mathrm{Pin}(V) \to O(V)$$

and by restriction we get a homomorphism

$$\rho \colon \mathrm{Spin}(V) \to SO(V).$$

Proposition 37. *The kernel of* $\rho \colon \mathrm{Pin}(V) \to O(V)$ *is* $\{\pm 1\}$*. Likewise, the kernel of* $\rho \colon \mathrm{Spin}(V) \to SO(V)$ *is* $\{\pm 1\}$*.*

Proof. Let $(e_1, \dots, e_n)$ be an orthonormal basis of V and suppose that $v = v_1 \cdots v_k \in \ker(\rho)$. We may write

$$v = \alpha_0 + e_1\alpha_1.$$

where α_0, α_1 are polynomials in $e_2, \dots, e_n$. Suppose $v \in \mathrm{Cl}^0(V)$. Then $\alpha_0 \in \mathrm{Cl}^0(V)$ and $\alpha_1 \in \mathrm{Cl}^1(V)$. Using the assumption $v \in \ker(\rho)$ we get $ve_1 = e_1 v$ so

$$e_1\alpha_0 - \alpha_1 = \alpha_0 e_1 + e_1\alpha_1 e_1 = e_1\alpha_0 + \alpha_1$$

Hence $\alpha_1 = 0$ and v itself is a polynomial in $e_2, \dots, e_n$. Proceeding by induction, we see that v does not contain any of $e_1, \dots, e_n$ and hence $v = \pm 1$.

Suppose on the other hand that $v \in \mathrm{Cl}^1(V)$. Then $ve_1 = -e_1 v$ by assumption and we have $\alpha_0 \in \mathrm{Cl}^1(V)$, $\alpha_1 \in \mathrm{Cl}^0(V)$. Then as above we get

$$e_1\alpha_0 - \alpha_1 = e_1\alpha_0 + \alpha_1$$

so that α_1 and we proceed again as above to see $v = 0$. It follows that this case does not occur. □

Since every element of $O(n)$ may be written as a product of n reflections (this is proven by diagonalizing an orthogonal matrix over $\mathbb{C}$ to put it into block diagonal form), the maps ρ are surjective. These arguments together with the previous proposition give:

Corollary 38. $\mathrm{Pin}(V) = \{v_1 \cdots v_k \mid k \leq n, \|v_i\| = 1\}$.

Proposition 39. *The groups* $\mathrm{Pin}(n)$ *and* $\mathrm{Spin}(n)$ *are compact Lie groups.*

Proof. The subset $\mathrm{Cl}^\times(n)$ of multiplicatively invertible elements in $\mathrm{Cl}(n)$ is an open subset of the finite-dimensional vector space $\mathrm{Cl}(n) \cong \mathbb{R}^{2^n}$. Because the product on $\mathrm{Cl}(n)$ is bilinear, it is smooth. The same holds for the inversion map on $\mathrm{Cl}^\times(n)$. It follows that $(\mathrm{Cl}^\times(n), \cdot, 1)$ is a Lie group.

We claim that $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ are compact subsets of $\mathrm{Cl}^\times(n)$. Because $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ are also subgroups, It then follows from the Closed Subgroup Theorem from Lie group theory that these are Lie subgroups. In particular, they are closed submanifolds of $\mathrm{Cl}^\times(n)$.

For each $k \geq 1$ we have a continuous map

$$\lambda_k \colon \underbrace{S^{n-1} \times \cdots \times S^{n-1}}_{k \text{ factors}} \to \mathrm{Pin}(n), \quad (v_1, \ldots, v_k) \mapsto v_1 \cdots v_k.$$

As $S^{n-1} \times \cdots \times S^{n-1}$ is compact, the image of λ_k is compact. By Corollary 38 $\mathrm{Pin}(n) = \bigcup_{k=1}^n \mathrm{im}(\lambda_k)$, is a finite union of compact sets and hence itself compact. An analogous argument applies to $\mathrm{Spin}(n)$. $\square$

The map $\rho\colon \mathrm{Pin}(n) \to O(n)$ is then a smooth map, being the restriction of the obviously smooth map $\mathrm{Cl}^\times(n) \to \mathrm{Aut}(\mathrm{Cl}(n))$, $v \mapsto (x \mapsto \alpha(v)xv^{-1})$.

Corollary 40. $\rho\colon \mathrm{Pin}(n) \to O(n)$ *and* $\rho\colon \mathrm{Spin}(n) \to SO(n)$ *are two-fold smooth coverings. In particular,* $\dim \mathrm{Pin}(n) = \dim O(n) = n(n-1)/2$ *and* $\dim \mathrm{Spin}(n) = \dim SO(n) = n(n-1)/2$.

Proof. It is enough to find an open neighborhood of $1 \in O(n)$ which is evenly covered by ρ. By Proposition 37 this amounts to finding an open neighborhood $U \subset \mathrm{Pin}(n)$ of $1 \in \mathrm{Pin}(n)$ so that $U \cap (-U) = \emptyset$. But this follows easily by the continuity of the map $v \mapsto -v$ on $\mathrm{Pin}(n)$. The case of $\mathrm{Spin}(n)$ follows immediately. The dimension computations follow from the corresponding computations for $O(n)$ and $SO(n)$ (which is a component of $O(n)$). $\square$

Example 41.

1. *We know that* $\mathrm{Cl}(1) = \mathbb{C}$. *It follows that* $\mathrm{Spin}(1) = \{v_1 \cdots v_{2k} \mid \|v_i\| = 1, v \in \mathbb{R}\} = \{\pm 1\}$ *and the map* $\mathrm{Spin}(1) \to SO(1) = \{1\}$ *is the constant map* 1.

2. *Next* $\mathrm{Cl}(2) = \mathbb{H}$. *Then* $\mathrm{Spin}(2) = \{v_1 \cdots v_{2k} \mid v_i \in \mathbb{R}^2, \|v_i\| = 1\}$. *Consider for* $|\alpha|^2 + |\beta|^2 = 1$ *and* $|x|^2 + |y|^2 = 1$ *the expression*

$$(\alpha i + \beta j)(xi + yj) = (-\alpha x - \beta y) + (\alpha y - \beta x)k =: \gamma_0 + \gamma_1 k.$$

Then again $|\gamma_0|^2 + |\gamma_1|^2 = 1$. *If we view* $(x, y) \in \mathbb{R}^2$ *as* $xi + yj$ *we see that* $\mathrm{Spin}(2) = S^1 \subset \mathbb{C}$, *using the isomorphism* $\langle 1, k\rangle_{\mathbb{R}} \cong \mathbb{C}$ *(sending* k *to* i*). The map* $\rho\colon \mathrm{Spin}(2) \to SO(2)$ *takes the following form.*

$$\begin{aligned}(\eta_0 + \eta_1 k)(x_1 i + x_2 j)(\eta_0 - \eta_1 k) &= (\eta_0 + \eta_1 k)^2 (x_1 i + x_2 j) \\ &= (\eta_0 + \eta_1 k)^2 (x_1 + x_2 k) i\end{aligned}$$

which shows that $\rho\colon S^1 \to S^1$ *may be identified with* $\rho(z) = z^2$.

3. *On the exercise sheet, we will see that* $\mathrm{Spin}(3) = S^3$ *so that* $SO(3) = \mathbb{R}P^3$. *In particular,* $\pi_1(SO(3)) = \mathbb{Z}/2$.

Proposition 42. *The Lie group* $\mathrm{Spin}(n)$ *is connected for* $n \geq 2$ *and simply-connected for* $n \geq 3$.

Proof. We have the long exact sequence for a covering

$$0 \to \pi_1(\mathrm{Spin}(n)) \to \pi_1(SO(n)) \to \pi_0(\mathbb{Z}/2) \to \pi_0(\mathrm{Spin}(n)) \to \pi_0(SO(n)) \to 0$$

The space $SO(n)$ is connected for all n. Moreover $\pi_1(SO(n)) = \mathbb{Z}/2$ for $n \geq 3$. For $n = 3$ this follows from $SO(3) = \mathbb{R}P^3$. The result now follows by induction: consider the long exact sequence for the fibration

$SO(n-1) \to SO(n) \to S^{n-1}$. Because S^{n-1} is simply-connected for $n > 3$ the result follows for $SO(n)$ as soon as it is known for $SO(n-1)$.

The map $\pi_0(\mathbb{Z}/2) \to \pi_0(\mathrm{Spin}(n))$ is the constant map, because $+1$ and -1 in $\mathrm{Spin}(n)$ may be joined by the path $(\cos(t)e_1 + \sin(t)e_2)\, e_1$ for $t \in [0, \pi]$. From the exact sequence it follows that $\pi_1(SO(n)) \to \pi_0(\mathbb{Z}/2)$ is a bijection and so the injection $\pi_1(\mathrm{Spin}(n)) \to \pi_1(SO(n))$ has zero image. It follows that $\pi_1(\mathrm{Spin}(n)) = 0$. $\square$

Corollary 43. *For $n \geq 3$ the map $\rho\colon \mathrm{Spin}(n) \to SO(n)$ is the universal covering of $SO(n)$.*

3.3 The Solution: Constructing Clifford module Bundles

We return to the problem of defining transition function $\psi_{ji}\colon U_{ij} \to \mathrm{Aut}(W)$, where W is a $\mathrm{Cl}(n)$-module, from given $\phi_{ji}\colon U_{ij} \to SO(n)$. Let us assume that we may lift the ϕ_{ji} along our covering map $\rho\colon \mathrm{Spin}(n) \to SO(n)$ to maps $\tilde{\phi}_{ji}\colon U_{ij} \to \mathrm{Spin}(n)$. We then attempt to define

$$\psi_{ji}(x)(w) = \tilde{\phi}_{ji}(x) \cdot w$$

in terms of the Clifford multiplication $\mathrm{Spin}(n) \times W \to W$. Then we have the compatibility

$$\phi_{ji}(x)(v) \cdot \psi_{ji}(x)(w) = \tilde{\phi}_{ji}(x) v \left(\tilde{\phi}_{ji}(x)\right)^{-1} \tilde{\phi}_{ji}(x) \cdot w = \tilde{\phi}_{ji}(x) \cdot vw = \psi_{ji}(x)(vw)$$

In order to get a well-defined vector bundle by the clutching construction, we however also need the cocycle condition, i.e. that the map

$$\sigma_{ijk} = \tilde{\phi}_{ik}^{-1} \cdot \tilde{\phi}_{ij} \cdot \tilde{\phi}_{jk}\colon U_{ijk} \to \mathrm{Spin}(n)$$

is always equal to one. Because our original transition functions ϕ_{ij} satisfy the cocycle condition and since the kernel of $\rho\colon \mathrm{Spin}(n) \to SO(n)$ is $\{\pm 1\}$ it follows that σ_{ijk} takes values in $\mathbb{Z}/2$. Note that for every $x \in U_{ij}$ there are two choices for $\tilde{\phi}_{ij}(x)$, since we are lifting along a two-fold covering.

Can we modify the maps $\tilde{\phi}_{ij}$ in a consistent way so that all σ_{ijk} become the constant map 1. This kind of *obstruction problem* is described by the Čech cohomology group $\check{H}^2(\{U_i\}; \mathbb{Z}/2)$.

Definition 44. *Let X be topological space with open cover $\mathcal{U} = (U_i)_{i \in I}$ indexes over a totally ordered set I, whose finite intersections $U_{ijk\cdots}$ are all empty or contractible (a so-called* good cover*). Define the Čech complex as follows. The group $\check{C}_n(\mathcal{U})$ is the free abelian group generated by all ordered tuples of indices $(i_0, \ldots, i_n)$ with $U_{i_0} \cap \cdots \cap U_{i_n} \neq \emptyset$. The differential is given on a basis by $\partial(i_0, \ldots, i_n) = \sum_{k=0}^{n} (-1)^k (i_0, \cdots, \widehat{i_k}, \cdots, i_n)$. The Čech co-complex with coefficients in an abelian group G is defined as $\check{C}^n(\mathcal{U}; G) = \mathrm{Hom}(\check{C}_n(\mathcal{U}); G)$.*

We then have the following Mayer-Vietoris principle:

Proposition 45. *The homology of $\check{C}_*(X)$ coincides with the singular homology of X. The cohomology of $\check{C}^*(X; G)$ coincides with the singular cohomology of X with coefficients in G.*

For a triangulated manifold this can be made concrete in the following way: Choose a triangulation of M^n and consider the dual cell decomposition of M (where k-cells are in one-to-one correspondence to the $(n-k)$-simplices on M). Let $(U_i)_{i \in I}$ be the covering of M where the U_i are thickenings of the top cells in this dual decomposition. This is a good cover of M and the Čech-complex associated to this covering is canonically isomorphic to the simplicial chain complex associated to the given triangulation. In the following we may work with a good cover of M of this sort.

Note that for a good cover, the maps σ_{ijk} considered before are constant, because they are defined on contractible sets. The elements $\sigma = \{\sigma_{ijk}\} \in \check{C}^2(\mathcal{U};\mathbb{Z}/2)$ define a Čech cocycle, i.e. satisfy $\delta\sigma = 1$:

$$
\begin{aligned}
&\sigma_{jkl}\sigma_{ikl}^{-1}\sigma_{ijl}\sigma_{ijk}^{-1}\\
=&\tilde{\phi}_{jl}^{-1}\tilde{\phi}_{jk}\tilde{\phi}_{kl}(\tilde{\phi}_{il}^{-1}\tilde{\phi}_{ik}\tilde{\phi}_{kl})^{-1}\tilde{\phi}_{il}^{-1}\tilde{\phi}_{ij}\tilde{\phi}_{jl}(\tilde{\phi}_{ik}^{-1}\tilde{\phi}_{ij}\tilde{\phi}_{jk})^{-1}\\
=&\tilde{\phi}_{jl}^{-1}\tilde{\phi}_{jk}\tilde{\phi}_{kl}\tilde{\phi}_{kl}^{-1}\tilde{\phi}_{ik}^{-1}\tilde{\phi}_{il}\tilde{\phi}_{il}^{-1}\tilde{\phi}_{ij}\tilde{\phi}_{jl}\tilde{\phi}_{jk}^{-1}\tilde{\phi}_{ij}^{-1}\tilde{\phi}_{ik}\\
=&\tilde{\phi}_{jl}^{-1}\tilde{\phi}_{jk}\tilde{\phi}_{ik}^{-1}\tilde{\phi}_{ij}\tilde{\phi}_{jl}\tilde{\phi}_{jk}^{-1}\tilde{\phi}_{ij}^{-1}\tilde{\phi}_{ik}\\
=&\tilde{\phi}_{jl}^{-1}(\tilde{\phi}_{jk}\tilde{\phi}_{ik}^{-1}\tilde{\phi}_{ij})\tilde{\phi}_{jl}\tilde{\phi}_{jk}^{-1}\tilde{\phi}_{ij}^{-1}\tilde{\phi}_{ik}\\
=&\tilde{\phi}_{jl}^{-1}\tilde{\phi}_{jl}\tilde{\phi}_{jk}^{-1}(\tilde{\phi}_{jk}\tilde{\phi}_{ik}^{-1}\tilde{\phi}_{ij})\tilde{\phi}_{ij}^{-1}\tilde{\phi}_{ik} = 1
\end{aligned}
$$

using that $\tilde{\phi}_{jk}\tilde{\phi}_{ik}^{-1}\tilde{\phi}_{ij}$ is central (it is ± 1 since it maps under ϕ to 1).

We obtain a Čech cohomology class $[\sigma] \in \check{H}^2(M;\mathbb{Z}/2)$. Suppose that $[\sigma] = 0$. Then we find $(\lambda_{ji}) \in \check{C}^1(\{U_i\};\mathbb{Z}/2)$ with $\delta\lambda = \sigma$ (so $(\delta\lambda)_{ijk} = \lambda_{jk}\lambda_{ik}^{-1}\lambda_{ij}$). Using λ we now redefine

$$\tilde{\phi}'_{ji} = \tilde{\phi}_{ji} \cdot \lambda_{ji}.$$

Then we get

$$\sigma'_{kji} = \sigma_{kji}\lambda_{ki}^{-1}\lambda_{kj}\lambda_{ji} = \sigma_{kji}(\delta\lambda)_{ikj} = 1.$$

The modified transition functions $\tilde{\phi}'_{ji}$ therefore satisfy the cocycle identity.

Definition 46. *The class $w_2(M) = [\sigma] \in H^2(M;\mathbb{Z}/2)$ is called the* second Stiefel-Whitney class *of M.*

It can be shown that the class of $w_2(M)$ is independent of the choice of open cover $\{U_i\}$. In summary, we have:

Proposition 47. *Let M be an oriented Riemannian manifold. Suppose $w_2(M) = 0$. Then we may consistently lift the transition functions of the tangent bundle TM to the group* $\mathrm{Spin}(n)$*. Using the* $\mathrm{Cl}(n)$*-module W we obtain an associated Clifford-module bundle $E \to M$ in the sense of Definition 28.*

(Note that if W has an invariant inner product, this induces also an inner product on the bundle E).

It remains to define a connection on E and to understand the $\mathrm{Cl}(n)$-modules W.

3.4 Interlude: Principal Bundles

We point out a more systematic view on the above construction. Let G be a Lie group.

Definition 48. *A G-principal bundle is a smooth fiber bundle $\pi\colon P \to M$ with a smooth, free G-action on P which preserves the fibers and acts fiber wise transitive on these.*

Example 49. *Let $E \to M$ be a rank k vector bundle. The* frame bundle *$P(E)_x = \{(b_1,\dots,b_k)$ basis of $E_x\} =$* $\mathrm{Iso}(\mathbb{R}^k, E_x)$*, $P(E) = \bigcup_{x\in M} P(E)_x \to M$, is a* $\mathrm{GL}_k(\mathbb{R})$*-principal bundle. The action of* $\mathrm{GL}_k(\mathbb{R})$ *on the fibers* $\mathrm{Iso}(\mathbb{R}^k, E_x)$ *is given by pre-composition.*

Suppose the vector bundle E is trivialized on an open cover U_i, so we have $E|_{U_i} \cong U_i \times \mathbb{R}^k$. Then E may be described in terms of the transition functions

$$\phi_{ji}\colon U_{ij} \to \mathrm{GL}_k(\mathbb{R}).$$

The frame bundle $P(E)$ may then be constructed by gluing the trivial bundles $U_i \times \mathrm{GL}_k(\mathbb{R})$ using the maps

$$\phi_{ji}\cdot(-)\colon U_{ij} \to \mathrm{Homeo}(\mathrm{GL}_k(\mathbb{R}), \mathrm{GL}_k(\mathbb{R}))$$

that take $x \in U_{ij}$ to the left multiplication map $\mathrm{GL}_k(\mathbb{R}) \to \mathrm{GL}_k(\mathbb{R})$ by the matrix $\phi_{ji}(x)$.

On the other hand, any $\mathrm{GL}_k(\mathbb{R})$-principal bundle has trivializations in which the transition functions are given by left multiplication of a certain matrix $\phi_{ji}(x)$. These may in turn be used to reconstruct a vector bundle E.

The formalism of $\mathrm{GL}_k(\mathbb{R})$-principal bundles and the formalism of rank k vector bundles are therefore equivalent points of view.

The existence of a Riemannian metric on E corresponds to a reduction of the structure group of $P(E)$ to $O(k)$. Indeed, if E has such a metric we may consider the $O(k)$-principal bundle $P_O(E)$ of orthonormal frames. On the other hand, such $O(k)$-valued transition may be used to define an inner product on E by using the standard inner product on the trivial pieces $U_i \times \mathbb{R}^k$. These then fit together since ϕ_{ji} is a fiber wise isometry. An *orientation* on E corresponds to a reduction to $SL_k(\mathbb{R})$ (resp. $SO(k)$ in the presence of a metric). Then we restrict to positively oriented frames (resp. positively oriented orthonormal frames) to construct principal bundles $P_{SL_k(\mathbb{R})}(E), P_{SO}(E)$.

The vector bundle E may be reconstructed as an associated bundle of $P(E)$:

$$E \cong P(E) \times_{\mathrm{GL}_k(\mathbb{R})} \mathbb{R}^k = (P(E) \times \mathbb{R}^k)/\sim$$

Here the equivalence relation '$\sim$' is given by $(\varphi g, v) \sim (\varphi g v)$ for $g \in \mathrm{GL}_k(\mathbb{R})$. Locally, the frame bundle reduces to $U \times \mathrm{GL}_k(\mathbb{R})$ and the associated bundle construction on $U \times \mathrm{GL}_k(\mathbb{R}) \times \mathbb{R}^k$ identifies $(x, \varphi A, v)$ with (x, φ, Av), so that the canonical map to $U \times \mathbb{R}^k = E|_U$ is an isomorphism.

Using this language, we have shown the following in the previous section:

Proposition 50. *We have $w_2(M) = 0$ precisely when we may consistently lift the $SO(n)$-valued transition functions to maps $\tilde{\phi}_{ji}$ to $\mathrm{Spin}(n)$. This is equivalent to the existence of a $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}}(TM) \to M$ along with a two-fold covering map $P_{\mathrm{Spin}}(TM) \to P_{SO}(TM)$ which is equivariant for the canonical covering map $\rho\colon \mathrm{Spin}(n) \to SO(n)$.*

In topological terms, this may be restated by saying that the classifying map $f\colon M \to BSO(n)$ lifts along the map $B\rho\colon B\,\mathrm{Spin}(n) \to BSO(n)$ to $M \to B\,\mathrm{Spin}(n)$. The map $B\rho$ has fiber $B\mathbb{Z}/2$, which implies that the obstruction class for such a lift is an element of $H^2(M; \pi_1(B\mathbb{Z}/2)) = H^2(M; \mathbb{Z}/2)$. This element may be identified with the second Stiefel-Whitney class we have constructed.

Definition 51. *Let M be an oriented smooth manifold. We call M a* spin manifold*, if $w_2(M) = 0$.*

It can be shown that if M is spin manifold, the Clifford bundle $E \to M$ from Proposition 47 is isomorphic to the associated bundle $P_{\mathrm{Spin}}(TM) \times_{\mathrm{Spin}(n)} W$, where $\mathrm{Spin}(n)$ acts on W by Clifford multiplication.

3.5 The Connection on the Clifford Bundle $E \to M$ and the corresponding Weitzenböck Formula

We want to turn the bundle $E \to M$ from Proposition 47 into a Dirac bundle. For this aim it remains to construct a connection on $E \to M$ compatible with the Levi-Civita connection on TM and the inner product on E, compare Definition 29.

From our construction (see Subsection 3.1) we have simultaneous orthogonal trivializations $U \times \mathbb{R}^n \cong TM|_U$ and $U \times W \cong E|_U$ (where $U \subset M$ is part of a fixed open cover of M used to define $E \to M$.)

The connection is defined as follows.

Choose a curve $\gamma\colon (-\varepsilon, \varepsilon) \to U$ through $p \in U$ and let $(e_1, \dots, e_n) \in T_pM$ be the standard basis in our trivialization. Using the Levi-Civita connection we may extend these to a parallel frame $(\bar{e}_1(t), \dots, \bar{e}_n(t))$ along γ. Choose matrices $A(t) \in SO(n)$, $A(0) = E_n$, with $A(t)e_i = \bar{e}_i(t)$. We wish to define the parallel transport of $w \in W = E_p$ along γ. For this we (uniquely) lift the map $A\colon (-\varepsilon, \varepsilon) \to SO(n)$ to a map $\tilde{A}\colon (-\varepsilon, \varepsilon) \to \mathrm{Spin}(n)$ with $\tilde{A}(0) = 1$.

Definition 52. *The connection on E is defined by declaring $\bar{w}(t) = \tilde{A}(t) \cdot w$ to be a parallel section of E along γ. We say this connection ∇^E is* induced *by the Levi-Civita connection.*

In local coordinates, this connection may be understood as follows (we will need this description later when applying the Weitzenböck formula to the present situation). Recall that the Christoffel symbols of the Levi-Civita connection are defined by

$$\nabla^{\mathrm{LC}}_{e_k}(e_i) = \Gamma^j_{ki} e_j.$$

We wish to compute $\nabla^E_{e_k}(w)$ for a 'constant section' $w \in W$. Choose a curve γ with $\gamma'(0) = e_k$.

$$A(t)e_i = \bar{e}_i(t) \Rightarrow e_i = C(t)\bar{e}_i(t), \qquad C(t) = A(t)^{-1}.$$

Then

$$\nabla^{\mathrm{LC}}_k e_i\big|_0 = \nabla_t|_{t=0}(C(t)\bar{e}_i(t)) = C'(0)\bar{e}_i(0) + C(0)\nabla_t|_{t=0}\bar{e}_i = C'(0)e_i$$

so that $C'(0)^j_i = \Gamma^j_{ki}$. Write

$$w = \tilde{C}(t) \quad \underbrace{\tilde{A}(t)w}_{\text{parallel for } \nabla^E}$$

for the lift $\tilde{C}$ of C to $\mathrm{Spin}(n)$ with $\tilde{C}(0) = 1$. Thus

$$\nabla^E_{e_k} w\big|_0 = \nabla_t|_{t=0}\left(\tilde{C}(t)\bar{w}(t)\right) = \tilde{C}'(0)\bar{w}(0) = \tilde{C}'(0)w$$

where $\tilde{C}'(0) \in T_1\,\mathrm{Spin}(n) \subset \mathrm{Cl}^0(n)$. It remains to compute $T_1\rho\colon T_1\,\mathrm{Spin}(n) \to T_1 SO(n)$, which maps $\tilde{C}'(0)$ to $C'(0)$. Since ρ is a covering map, the map $T_1\rho$ is an isomorphism.

The group $\mathrm{Spin}(n) \subset \mathrm{Cl}^0(n)$ is a submanifold of $\mathrm{Cl}^0(n)$. Thus $T_1\,\mathrm{Spin}(n)$ may be viewed as a linear subspace of $\mathrm{Cl}^0(n)$:

Proposition 53. *The set e_ie_j, $i < j$ is a basis of the vector subspace $T_1\,\mathrm{Spin}(n) \subset \mathrm{Cl}^0(n)$. We have*

$$T_1\rho(e_ie_j) = 2\begin{pmatrix} \ddots & & & & \\ & 0 & & -1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & \ddots \end{pmatrix}$$

where $+1$ is placed in row j, column i.

Proof. Consider the curve $\gamma(t) = \cos(t) + \sin(t)e_ie_j = (\sin(t)e_i - \cos(t)e_j)e_j$ inside $\mathrm{Spin}(n)$. It represents the tangent vector $\gamma'(0) = e_ie_j$. This shows that all e_ie_j belong to $T_1\,\mathrm{Spin}(n)$, which has dimension $n(n-1)/2$ so we have found a basis. For $x \in \mathbb{R}^n$ we compute

$$\frac{d}{dt}\bigg|_0 \rho\gamma(t)(x) = \frac{d}{dt}\bigg|_0 \gamma(t)x\gamma(t)^{-1} = e_ie_jx - xe_ie_j$$

Clearly, $e_ie_jx - xe_ie_j$ is zero for $x = e_k$, $k \neq i, j$. Also $e_ie_je_i - e_ie_ie_j = 2e_j$ for $x = e_i$ and similarly for $x = e_j$. □

We now rewrite $C'(0) = (\Gamma^j_{ki}) = \sum_{i<j} \Gamma^j_{ki} P_{ij}$ to conclude

$$\nabla^E_{e_k} w = \frac{1}{2}\sum_{i<j} \Gamma^j_{ki} e_ie_j w \tag{14}$$

for an orthonormal frame $(e_1, \ldots, e_n)$. We now compute the curvature. For this we work with an orthonormal frame $(e_1, \ldots, e_n)$ which is synchronous at p. In particular all Christoffel symbols vanish at p and at p these vector fields commute as $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i$ vanishes at p. In such a frame, the formula for the curvature K^{TM} at p simplifies to $K^i_{ljk}(p) = \nabla_{e_j}\Gamma^i_{kl} - \nabla_{e_k}\Gamma^i_{jl}$ (here ∇ just denotes directional derivatives of scalar valued functions). At the point p we then have

$$\begin{aligned}
2K^E(e_j, e_k)(p) &= \nabla^E_j \nabla^E_k - \nabla^E_k \nabla^E_j \overset{(14)}{=} \sum_{\alpha<\beta} \left(\nabla^E_j (\Gamma^\beta_{k\alpha} e_\alpha e_\beta) - \nabla^E_k (\Gamma^\beta_{j\alpha} e_\alpha e_\beta) \right) \\
&= \sum_{\alpha<\beta} \left(\nabla_j \Gamma^\beta_{k\alpha} - \nabla_k \Gamma^\beta_{j\alpha} \right) e_\alpha e_\beta = \sum_{\alpha<\beta} K^\beta_{\alpha jk} e_\alpha e_\beta = \sum_{\alpha<\beta} \langle K(e_j, e_k) e_\alpha, e_\beta \rangle e_\alpha e_\beta
\end{aligned}$$

The third equality uses the fact that the frame $(e_1, \dots, e_n)$ is synchronous at p.

Remark 54. *It is no surprise that $K^E(e_j, e_k) = \frac{1}{2} \sum_{\alpha<\beta} K^\beta_{\alpha jk} e_\alpha e_\beta$. Let*

$$P_{ij} = \begin{pmatrix} \ddots & & & & \\ & 0 & & -1 & \\ & & \ddots & & \\ & 1 & & 0 & \\ & & & & \ddots \end{pmatrix}$$

be the matrices from Proposition 53. Then the image of this element under the isomorphism $T_1\rho$ is

$$T_1\rho(K^E(e_j, e_k)) = T_1\rho \left(\frac{1}{2} \sum_{\alpha<\beta} K^\beta_{\alpha jk} e_\alpha e_\beta \right) = \sum_{\alpha<\beta} K^\beta_{\alpha jk} P_{\alpha\beta} = K^{TM}(e_j, e_k)$$

Since the connection on E is induced from the bundle $P_{\mathrm{Spin}}(TM)$, as is the connection on TM, this result can also be obtained from the general theory of connections on principal bundles.

We now examine the curvature term in the Weitzenböck formula $D^2 = \Delta + K$, where $K(s) = \sum_{j<k} e_j e_k K^E(e_j, e_k) s$, see Theorem 33. In our case

$$\begin{aligned}
\sum_{j<k} e_j e_k K^E(e_j, e_k) &= \frac{1}{2} \sum_{j,k} e_j e_k K^E(e_j, e_k) = \frac{1}{8} \sum_{j,k,\alpha,\beta} e_j e_k \langle K^{TM}(e_j, e_k) e_\alpha, e_\beta \rangle e_\alpha e_\beta \\
&= \frac{1}{8} \sum_\beta \Bigg(\frac{1}{3} \sum_{j,k,\alpha \text{ distinct}} \langle K^{TM}(e_j, e_k) e_\alpha + K^{TM}(e_k, e_\alpha) e_j + K^{TM}(e_\alpha, e_j) e_k, e_\beta \rangle e_j e_k e_\alpha \\
&\qquad + \sum_{j,k,(\alpha=j)} \langle K^{TM}(e_j, e_k) e_j, e_\beta \rangle e_j e_k e_j + \sum_{j,k,(\alpha=k)} \langle K^{TM}(e_j, e_k) e_k, e_\beta \rangle e_j e_k e_k \Bigg) e_\beta
\end{aligned}$$

Leaving β fixed, we have used here the anti-symmetry of K^{TM} to reduce the three-fold sum over (j, k, α) to the case $j \neq k$. The remaining cases (j, k, α) pairwise disjoint, $j = \alpha$, and $k = \alpha$ were then gathered as individual summands. The first summand consists of three equal parts. It vanishes by the Bianchi identity. By replacing j with k in the last summand, we see that the last two summands are equal. The above expression therefore reduces to (using $e_j e_k e_j e_\beta = e_k e_\beta$)

$$\frac{1}{4} \sum_{j,k,\beta} \langle K^{TM}(e_j, e_k) e_j, e_\beta \rangle e_k e_\beta = -\frac{1}{4} \sum_{k,\beta} \mathrm{Ric}(e_k, e_\beta) e_k e_\beta = \frac{1}{4} \mathrm{scal}_g.$$

Combined with Theorem 33 these calculations show:

Theorem 55. *Let (M, g) be an oriented Riemannian manifold with $w_2(M) = 0$. Let W be a $\mathrm{Cl}(n)$-module with compatible inner product and let $E \to M$ be the corresponding Dirac bundle with Dirac operator D. Then we have*

$$D^2 = \Delta + \frac{1}{4}\mathrm{scal}_g,$$

where scal_g operates on sections by scalar multiplication.

Remark 56. *The Dirac bundle $\mathrm{Cl}(TM) \cong \Lambda^*(TM) \cong \Lambda^*(T^*M)$ from Section 2.5. does not follow this construction scheme (that started in Section 3.1). In particular, the curvature term appearing there is different: On one forms it is given by the Ricci endomorphism and not by multiplication with the scalar curvature function.*

The following will be shown on exercise sheet 5:

Proposition 57. *In even dimensions $n = 2k$ there exists a unique $\mathrm{Cl}(n)$-representation Δ of complex dimension 2^k.*

We therefore obtain a canonical Dirac operator D on the vector bundle $S = P_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}} \Delta$. This is *the Dirac operator* on an even-dimensional spin manifold.

Remark 58. *The construction of the associated bundle E from the last sections may be carried out as soon as one has a $\mathrm{Spin}(n)$-representation W.*

3.5.1 Spinor Dirac Operator

As a $\mathrm{Spin}(n)$-module the representation Δ splits into two irreducible parts $\Delta = \Delta_+ \oplus \Delta_-$. The representations $\Delta_\pm$ are inequivalent irreducible representation of $\mathrm{Spin}(n)$. There are no other irreducible representations of $\mathrm{Spin}(n)$ so that $-1 \in \mathrm{Spin}(n)$ acts as multiplication with -1, see exercise sheet 6 [2]

Hence $S = S_+ \oplus S_-$ with induced connections $\nabla^{S_\pm}$. The Clifford multiplication with an element $v \in \mathbb{R}^n$ takes S_+ to S_- (and S_- to S_+). We may therefore view the Dirac operators as

$$D\colon C^\infty(S_\pm) \to C^\infty(S_\mp),$$

the so-called $\mathbb{Z}/2$-*graded Dirac operator* or the *Spinor Dirac Operator*. This splitting only exists on even-dimensional manifolds. For odd-dimensional manifolds, the Dirac operator does not split.

4 Linear Analysis on Manifolds

4.1 Linear Differential Operators

Recall the notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha \in \mathbb{N}^n$.

Definition 59. *Let $E, F \to M$ be vector bundles on a smooth manifold M^n of respective ranks $\mathrm{rk}E = p, \mathrm{rk}F = q$. A differential operator from $C^\infty(E)$ to $C^\infty(F)$ of order $\leq k$ is a linear map*

$$P\colon C^\infty(E) \to C^\infty(F).$$

In local coordinates $(x^1, \dots, x^n)$ on $U \subset M$ and in local trivializations $E|_U = U \times \mathbb{R}^p$, $F|_U = U \times \mathbb{R}^q$ we require that P may be expressed in the form

$$(P\varphi)(x) = \sum A^\alpha(x) \frac{\partial^{|\alpha|}\varphi}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}, \qquad x \in U,$$

where $A^\alpha\colon U \to \mathbb{R}^{q \times p}$ are smooth functions (this condition is independent of the choice of trivializations and coordinates).

[2] This is not correct: Exercise Sheet 6 indeed yields a $\mathrm{Spin}(n)$-representation which extends to an algebra representation of $\mathrm{Cl}(n)^0$. But this "linearizable" representation does in general not coincide with the given $\mathrm{Spin}(n)$-representation. An example for $n = 2$ is given by the irreducible complex $\mathrm{Spin}(2) = S^1$-representation of weight 3 ("Spin = 3/2"). Then the resulting S^1-representation is of weight -1.

Letting $\mathcal{D}_k(E,F)$ denote the set of differential operators from $C^\infty(E)$ to $C^\infty(F)$ of order $\leq k$, we have inclusions

$$\mathcal{D}_k(E,F) \supset \mathcal{D}_{k-1}(E,F) \supset \cdots \supset \mathcal{D}_0(E,F) = C^\infty(\mathrm{Hom}(E,F))$$

For $P \in \mathcal{D}_k$, $Q \in \mathcal{D}_l$ we have $Q \circ P \in \mathcal{D}_{k+l}$ for the composite.

Example 60. *On a Riemannian manifold (M,g) let $E = M \times \mathbb{R}$, so $C^\infty(E) = C^\infty(M,\mathbb{R})$ are the real-valued functions on M. Let $F = TM$ be the tangent bundle. From coordinates (x^i) on M we get a local frame $(\partial_i = \partial/\partial x^i)_i$ of F. Recall that the* gradient *of a function $f \in C^\infty(M)$ is the unique vector field $\mathrm{grad}_g(f)$ satisfying*

$$df = g(\mathrm{grad}_g(f), -).$$

We thus get a linear map $\mathrm{grad}_g \colon C^\infty(M) \to C^\infty(TM)$. Locally,

$$\mathrm{grad}_g(f) = g^{ij}\frac{\partial f}{\partial x^i}\partial_j$$

so that $\mathrm{grad}_g \in \mathcal{D}_1(E,F)$. We have (where '1' appears in the i-th position)

$$A^{(0,\cdots,0,1,0,\cdots,0)}(x) = \begin{pmatrix} g^{i1}(x) \\ \vdots \\ g^{in}(x) \end{pmatrix},$$

all other A^α vanish.

Example 61. *The exterior derivative $d \colon C^\infty(\Lambda^k T^*M) \to C^\infty(\Lambda^{k+1}T^*M)$ is a differential operator of first order.*

Example 62. *For a Dirac bundle S the corresponding Dirac operator $D \in \mathcal{D}_1(S,S)$ is also differential operator of first order.*

Example 63. *For a vector bundle $E \to M$ with connection ∇^E over a Riemannian manifold we have seen the connection Laplacian $\Delta \in \mathcal{D}_2(E,E)$, which is of second order.*

Example 64. *Let (M,g) be a Riemannian manifold, $E = TM$, $F = M \times \mathbb{R}$. The divergence of $X \in C^\infty(TM)$ is the function $\mathrm{div}X = -d^*\alpha = *d(*\alpha)$, where $\alpha = X^\flat$ is the differential form which is dual (in the sense of the metric g) to the vector field X (note that [Roe] uses the opposite convention $\mathrm{div}X = +d^*$).*

Write $\alpha = A_i dx^i$. Then $*\alpha = \sum_{i,j}(-1)^{j+1}A_i\sqrt{g}g^{ij}dx^1 \wedge \ldots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \ldots dx^n$, where $g = \det(g_{ij})$ and so

$$d(*\alpha) = \partial_j\left(A_i g^{ij}\sqrt{g}\right)dx^1 \wedge \ldots \wedge dx^n$$

or

$$\begin{aligned} *d*\alpha &= \frac{1}{\sqrt{g}}\partial_j\left(A_i g^{ij}\sqrt{g}\right) \\ &= g^{ij}\partial_j A_i + A_i\partial_j g^{ij} + A_i g^{ij}\partial_j(\log\sqrt{g}) \end{aligned}$$

We have

$$\partial_j g^{ij} = -\Gamma^j_{ja}g^{ai} - \Gamma^i_{ja}g^{aj}$$

while

$$\partial_j g = g^{ab}g\partial_j(g_{ab})$$

using that $\frac{d}{dt}\det(A(t)) = \mathrm{tr}\left(\mathrm{Adj}(A(t))\frac{dA}{dt}\right)$ for every smooth function $t \mapsto A(t) \in \mathbb{R}^{n\times n}$ (Jacobi's formula). Therefore

$$\partial_j \log\sqrt{g} = -\frac{1}{\sqrt{g}}\frac{1}{2\cdot\sqrt{g}}\partial_j g = \Gamma^a_{ja}$$

Hence we have the following local expression for the divergence

$$-d^*\alpha = g^{ij}\partial_j A_i - A_i g^{jk}\Gamma^i_{jk}.$$

The differential operator is therefore described by the matrices

$$A^{(0,\dots,0)} = (-g^{jk}\Gamma^1_{jk}, \dots, -g^{jk}\Gamma^n_{jk})$$

and (where '1' stands in the j-th place)

$$A^{(0,\dots,1,\dots,0)} = (g^{1j}, \dots, g^{nj})$$

Example 65. *In local coordinates where $X = X^k\partial_k$, we may also write*

$$\mathrm{div}(X) = \frac{1}{\sqrt{g}}\partial_k\left(\sqrt{g}X^k\right). \tag{15}$$

For the proof, note that $*\langle X, -\rangle = \iota_X \mathrm{dvol}$ (obvious for $X = e_1$ in an orthonormal frame). This means

$$*X^\flat = \sum_k (-1)^{k-1} X^k \sqrt{g}\, dx^{1\cdots\hat{k}\cdots n}$$

and so

$$d * X^\flat = \partial_k(\sqrt{g}X^k)dx^{1\cdots n}$$

This n-form is clearly Hodge dual to (15). This local expression for the divergence is similar to that on $\mathbb{R}^n$. Expanding $\partial_k(\sqrt{g})$ similarly as above leads to

$$\mathrm{div}(X) = \partial_k X^k + \Gamma^a_{ka}X^k.$$

Definition 66. *Let (M, g) be a Riemannian manifold with volume element $\mathrm{dvol}_g = \sqrt{g}dx^1\cdots dx^n$ (note that an orientation of M is not required here, so dvol is regarded as a measure on M.) Suppose that the vector bundle E is equipped with an inner product $(-,-)_E$. We define the L^2-*inner product

$$\langle\varphi, \psi\rangle_E = \int_M (\varphi(x), \psi(x))_E \,\mathrm{dvol}_g$$

for $\varphi, \psi \in C^\infty(E)$, where at least one of these is required to have compact support.

This defines an inner product (a positive definite symmetric bilinear form / Hermitian form) on the space of sections $C^\infty(E)$.

Proposition 67. *Suppose $E, F \to M$ are vector bundles with an inner product over a Riemannian manifold M. For all $P \in \mathcal{D}_k(E, F)$ there exists a unique $P^* \in \mathcal{D}_k(F, E)$ with the property*

$$\langle Pu, v\rangle_F = \langle u, P^*v\rangle_E \tag{16}$$

for all $u \in C^\infty(E)$ and $v \in C^\infty(F)$, one of which is required to have compact support.

Definition 68. *The unique differential operator P^* described in Proposition 67 is called the* formal adjoint *of the differential operator P.*

Proof. Since the inner product is positive definite, it is clear that there exists at most one such operator P^*. We wish to prove that it is a differential operator of order $\leq k$. For this calculate in local coordinates.

Suppose therefore that the supports of u, v are both contained in a coordinate neighborhood U of M. Choose orthonormal frames of $E|_U, F|_U$. Then

$$\begin{aligned}
\int_U (Pu, v)_F \mathrm{dvol} &= \sum_{|\alpha|\le k} \int_U \left(A^\alpha \frac{\partial^{|\alpha|} u}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}, v \right)_F \sqrt{g} dx \\
&= \sum_{|\alpha|\le k} \int_U \left(\frac{\partial^{|\alpha|} u}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}, \sqrt{g}(A^\alpha)^* v \right)_E dx \\
&= \sum_{|\alpha|\le k} \int_U (-1)^{|\alpha|} \left(u, \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}} \left(\sqrt{g}(A^\alpha)^* v\right) \frac{1}{\sqrt{g}} \right) \mathrm{dvol}
\end{aligned}$$

where we have used integration by parts. It follows that

$$P^* v = \frac{1}{\sqrt{g}} \sum_{|\alpha|\le k} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}(\sqrt{g}(A^\alpha)^* v)}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}$$

This shows that P^* is a differential operator of order $\le k$. To prove (16) for u, v arbitrary, where we assume without loss of generality that the support of u is compact, we proceed as follows. Write $u = u_1 + \cdots + u_l$ where u_j has support in some coordinate neighborhood U_j. Then we write $v = v_1 + \cdots + v_l + \bar{v}$ where the support of v_j is within U_j and where the support of $\bar{v}$ is disjoint from all the U_j. Then (16) follows from linearity and locality of P, P^*. □

Example 69. *If $P = A \in C^\infty(\mathrm{Hom}(E, F))$, the formal adjoint is the ordinary adjoint of the family of linear maps $E_x \to F_x$.*

Example 70. *Let (M^n, g) be an oriented Riemannian manifold. Consider $d \colon \Omega_c^{k-1}(M) \to \Omega_c^k(M)$. The formal adjoint of d is the operator d^* defined above in equation (12). This follows from Stokes' Theorem for $\alpha \in \Omega^k$, $\beta \in \Omega^{k-1}$*

$$\begin{aligned}
0 = \int_M d(\beta \wedge *\alpha) &= \int_M d\beta \wedge *\alpha + (-1)^{k-1} \int_M \beta \wedge d(*\alpha) \\
&= \langle d\beta, \alpha \rangle + (-1)^{k-1+(n-k+1)n+n-k+1} \int_M \beta \wedge ** d(*\alpha) \\
&= \langle d\beta, \alpha \rangle - \langle \beta, d^* \alpha \rangle
\end{aligned}$$

using the definition of the Hodge star $d\beta \wedge *\alpha = (d\beta, \alpha)\mathrm{dvol}_g$, where $\mathrm{dvol}_g \in \Omega^n(M)$ is the (oriented) volume form on M.

Example 71. *Let (M, g) be a closed Riemannian manifold and let $S \to M$ be a bundle with connection and compatible inner product. The covariant derivative is a map $\nabla \colon C^\infty(S) \to C^\infty(T^*M \otimes S)$. We shall compute the formal adjoint ∇^*. Choose a frame $(e_1, \ldots, e_n)$ of TM on U with corresponding dual frame ε^i, so that $\varepsilon^i(e_j) = \delta^i_j$. Let $\varepsilon^i \otimes s_i \in C^\infty(T^*M \otimes S)$ and $s \in C^\infty(S)$. Define $\omega = \varepsilon^i (s_i, s)_S \in \Omega^1_c(U)$ using the inner product on S. Then by Stokes' Theorem and Lemma 34 we have*

$$\begin{aligned}
0 &= \int_M d^*\omega \\
d^*\omega &= g^{jk} e_j \lrcorner \nabla_{e_k} \omega \\
\nabla_{e_k} \omega &= -\Gamma^i_{kq} \varepsilon^q (s_i, s) + \varepsilon^i \nabla_{e_k} (s_i, s)
\end{aligned}$$

so

$$0 = \int_M \left(g^{jk} \Gamma^i_{kj} (s_i, s) - g^{jk} \nabla_{e_k} (s_j, s) \right) \mathrm{dvol} = \int_M \left(g^{jk} \Gamma^i_{kj} (s_i, s) - g^{jk} (\nabla_k s_j, s) - g^{jk} (s_j, \nabla_k s) \right) \mathrm{dvol}.$$

Since $\nabla(s) = \varepsilon^k \nabla_k(s)$ *we also have*

$$\big(\varepsilon^j \otimes s_j, \nabla(s)\big) = g^{jk}(s_j, \nabla_k s)$$

Integrating over M *then gives*

$$\langle \varepsilon^j \otimes s_j, \nabla(s)\rangle = \int_M \big(g^{jk}\Gamma^i_{kj} s_i - g^{jk}\nabla_k s_j, s\big)\, d\mathrm{vol}.$$

Thus $\nabla^*(\varepsilon^i \otimes s_i) = g^{jk}\Gamma^i_{kj} s_i - g^{jk}\nabla_k s_j$. *A similar computation can be used to give an alternative approach to Example 64.*

In case $(e_1, \dots, e_n)$ *is an orthonormal frame, synchronous at* p, *this simplifies to*

$$\nabla^*(\varepsilon^i \otimes s_i)\big|_p = -\sum_{i=1}^n \nabla_i(s_i)$$

From this calculation, we obtain for the connection Laplacian defined in (11):

Proposition 72. $\Delta = \nabla^* \circ \nabla$.

Proof. For $(e_1, \dots, e_n)$ an orthonormal frame, synchronous at p we have $\Delta(s)\big|_p = -\sum \nabla_i^2(s)$, while

$$\nabla^*\nabla(s)|_p = \nabla^*(\varepsilon^i \nabla_i s) = -\sum \nabla_i^2(s).$$ □

It follows that Δ is a non-negative operator:

$$\langle \Delta(s), s\rangle = \langle \nabla(s), \nabla(s)\rangle \geq 0.$$

Using $D^2 = \Delta + K$ from Theorem 33 we conclude:

Theorem 73 (Bochner)**.** *Let* $S \to M$ *be a Dirac bundle with* $\langle Ks, s\rangle > 0$ *for all* $s \neq 0$. *Then* $\ker(D) = 0$.

Proof. $Ds = 0 \Rightarrow D^2 s = 0 \Rightarrow 0 = \langle D^2 s, s\rangle = \langle \Delta s, s\rangle + \langle Ks, s\rangle$ is the sum of two non-negative numbers. Hence both are zero, so $Ks = 0$, which by assumption implies $s = 0$. □

Example 74. *Let* (M, g) *be a spin manifold, let* W *be a* $\mathrm{Cl}(n)$*-representation, and let* $S \to M$ *be the corresponding Dirac bundle. If* $\mathrm{scal}_g > 0$ *at every point, then* $\ker(D) = 0$. *This follows since in this case,* K *is multiplication with* $\frac{1}{4}\mathrm{scal}_g$ *(see Theorem* 55*).*

Example 75. *Let* $S \to M$ *be a Dirac bundle. Then* $D^* = D$ *(so the Dirac operator is* formally self-adjoint*). To prove this, we choose an orthonormal frame* $(e_1, \dots, e_n)$ *synchronous at* p. *At* p *we have*

$$\begin{aligned}(Ds_1, s_2)|_p - (s_1, Ds_2)|_p &= \sum_i \big((e_i\nabla_i s_1, s_2)_S - (s_1, e_i\nabla_i s_2)\big)_S \\ &= \sum_i (\nabla_i(e_i s_1), s_2)_S + (e_i s_1, \nabla_i s_2)_S \\ &= \sum_i \nabla_i\, (e_i s_1, s_2)_S = d^*\omega,\end{aligned}$$

where $\omega = -(e_i s_1, s_2)\varepsilon^i$ *(use Lemma* 34*). From Stokes' Theorem it follows that* $\langle Ds_1, s_2\rangle = \langle s_1, Ds_2\rangle$.

4.2 Sobolev Spaces

Definition 76. *Two norms* $\|\cdot\|_1, \|\cdot\|_2$ *on a (possible infinite dimensional) vector space* V *are* equivalent *if we find constants* $C_1, C_2 > 0$ *so that*

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1$$

for all $x \in V$.

In the following we repeatedly use the following principle: Let $\|-\|_1$ and $\|-\|_2$ be two equivalent norms on V. Let $\overline{(V, \|-\|_1)}$ and $\overline{(V, \|-\|_2)}$ be the completions of V with respect to these norms (consisting of equivalence classes of Cauchy sequences). Then the identity induces linear bounded maps

$$\overline{(V, \|-\|_1)} \rightleftarrows \overline{(V, \|-\|_2)}$$

and hence these two completions are canonically isomorphic as topological vector spaces.

Let $E \to (M, g)$ be a vector bundle with Hermitian metric over a Riemannian manifold (M, g). Recall that for $u, v \in C_c^\infty(E)$ we have defined the L^2-inner product as

$$\langle u, v\rangle_{L^2} = \int_M (u, v)_E \, d\mathrm{vol}.$$

This defines a Hermitian form on the space $C_c^\infty(E)$.

Definition 77. $L^2(E)$ *is the Hilbert space completion of the inner product space* $(C_c^\infty(E), \langle -, -\rangle_{L^2})$. *We call it the* Hilbert space of L^2-sections of E (*even though, strictly speaking, the elements are not genuine sections, but may be modified on sets of measure zero*). *We write* $\|u\|_{L^2}^2 = \langle u, u\rangle_{L^2}$, *which controls the average of* u.

We wish to define norms which control not only the average of the values of u, but also all averages of the derivatives of order up to k. We begin with the important special case of the torus $M = T^n = \mathbb{R}/(2\pi\mathbb{Z}^n)$ with the standard flat metric (induced from $\mathbb{R}^n$) and the trivial bundle $E = M \times \mathbb{C}$. Then $C^\infty(E) = C^\infty(M, \mathbb{C})$ are 2π-periodic complex-valued functions.

Fourier Expansions. The functions $u_\nu(x) = (2\pi)^{-n/2} e^{i\langle\nu, x\rangle}$ for $\nu \in \mathbb{Z}^n$ form an orthonormal basis of $L^2(T^n, \mathbb{C})$ (they are obviously orthonormal. Using the Stone-Weierstraß Theorem, they can be seen to span the Hilbert space). It follows that any $\varphi \in C^\infty(T^n)$ may be expressed as an L^2-convergent series

$$\varphi = \sum_{\nu \in \mathbb{Z}^n} \hat{\varphi}(\nu) u_\nu, \qquad \text{where} \quad \hat{\varphi}(\nu) = \langle \varphi, u_\nu\rangle = \int_{T^n} \varphi(x)\overline{u_\nu(x)} dx.$$

We call $\{\hat{\varphi}(\nu)\}_{\nu \in \mathbb{Z}^n}$ the *Fourier coefficients* of φ. Then we have *Parseval's identity*

$$\|\varphi\|_{L^2}^2 = \sum_{\nu, \mu} \hat{\varphi}(\nu)\overline{\hat{\varphi}(\mu)}\langle u_\nu, u_\mu\rangle = \sum_\nu |\hat{\varphi}(\nu)|^2.$$

Differentiation and multiplication correspond to each other under the Fourier transform:

$$\widehat{\frac{\partial\varphi}{\partial x^j}}(\nu) = \left\langle \frac{\partial \varphi}{\partial x^j}, u_\nu \right\rangle = -\left\langle \varphi, \frac{\partial u_\nu}{\partial x^j}\right\rangle = i\nu_j \hat{\varphi}(\nu), \tag{17}$$

using integration by parts and $\frac{\partial u_\nu}{\partial x^j} = i\nu_j u_\nu$.

This relationship between differentiation and the Fourier coefficients leads us to the following definition:

Definition 78. *Let* $k \in \mathbb{N}$ *and* $u, v \in C^\infty(T^n)$. *Define the* k*-th Sobolev norm* (*also called* H^k *or* $W^{k,2}$)

$$\langle u, v\rangle_{W^k} = \sum_{\nu \in \mathbb{Z}^n} \hat{u}(\nu)\overline{\hat{v}(\nu)}(1 + \|\nu\|^2)^k$$

Using (17), *this series is seen to converge absolutely. The Hilbert space completion* $W^k(T^n)$ *of* $(C^\infty(T^n), \langle -, -\rangle_{W^k})$ *is called the* Sobolev space *of degree* k.

For example, $W^0(T^n) = L^2(T^n)$.

Proposition 79. *If $0 < k_1 < k_2$, then the inclusion $W^{k_2}(T^n) \to W^{k_1}(T^n)$ (given by extending the uniformly continuous map $C^\infty(T^n) \to W^{k_1}(T^n)$ to the completion $W^{k_2}(T^n)$) is continuous.*

The map $F\colon W^{k_2}(T^n) \to W^{k_1}(T^n)$ is injective (this is not obvious, because a non-injective map may well be injective on a dense subspace). It suffices to show that $W^k \to L^2$ is injective, because then F may be post-composed with $W^{k_1} \to L^2$ to give the injective map $W^{k_2} \to L^2$. Suppose therefore that $u \in W^{k_2}$ satisfies $\|u\|_{L^2} = 0$. Then all Fourier coefficients $\hat{u}(\nu) = 0$ vanish and so $\|u\|_{W^{k_2}} = 0$ and $u = 0$.

Example 80. *Let $\varphi \in C^\infty(T^n)$. Then by (17) we have*

$$\|\varphi\|_{W^1}^2 = \sum_{\nu \in \mathbb{Z}^n} |\hat{\varphi}(\nu)|^2 (1 + \|\nu\|^2) = \|\varphi\|_{L^2}^2 + \|\mathrm{grad}(\varphi)\|_{L^2}^2.$$

Thus $\|\cdot\|_{W^1}$ controls both φ and its derivative in the average. Similarly, $\|\cdot\|_{W^k}$ controls all derivatives up to order k in the average.

Definition 81. *For $k \in \mathbb{N}$ and $\varphi \in C^\infty(T^n)$ let*

$$\|\varphi\|_{C^k} = \max_{|\alpha| \le k} \sup_{x \in T^n} \left| \frac{\partial^{|\alpha|}\varphi}{\partial^{\alpha_1}x^1 \cdots \partial^{\alpha_n}x^n}(x) \right|.$$

This norm controls the mixed derivatives up to order k at every point (not *just in the average*)*. Note however, that this norm is* not *induced by an inner product.*

The completion of the normed space $(C^\infty(T^n), \|\cdot\|_{C^k})$ may be identified with the Banach space $C^k(T^n)$. The follows from the fact that any $f \in C^k(T^n)$ may be approximated in the C^k-norm by smooth functions.

Proposition 82. *The identity map $(C^\infty(T^n), \|\cdot\|_{C^k}) \to (C^\infty(T^n), \|\cdot\|_{W^k})$ is continuous. We thus get a continuous map $C^k(T^n) \to W^k(T^n)$ on the completions.*

Proof. Using the multinomial theorem we calculate for $u \in C^\infty(T^n)$:

$$\|u\|_{W^k}^2 = \sum_{\nu \in \mathbb{Z}^n} |\hat{u}(\nu)|^2 (1 + \|\nu\|^2)^k = \sum_{\nu \in \mathbb{Z}^n} |\hat{u}(\nu)|^2 \sum_{|\alpha| \le k} \binom{k}{|\alpha|}\binom{|\alpha|}{\alpha} \|\nu^\alpha\|^2$$

By Parseval's Theorem and (17) we have

$$\sum_{\nu \in \mathbb{Z}^n} |\hat{u}(\nu)\nu^\alpha|^2 = \left\| \frac{\partial^{|\alpha|}u}{\partial x^\alpha} \right\|_{L^2}^2.$$

Combining these two equations with $\left\| \frac{\partial^{|\alpha|}u}{\partial x^\alpha} \right\|_{L^2}^2 \le \mathrm{vol}(T^n)\|u\|_{C^k}^2$ completes the proof. ☐

As by-product of the proof we obtain the following generalization of Example 80:

$$\|u\|_{W^k}^2 = \sum_{|\alpha| \le k} \binom{k}{|\alpha|}\binom{|\alpha|}{\alpha} \left\| \frac{\partial^{|\alpha|}u}{\partial x^\alpha} \right\|_{L^2}^2 \tag{18}$$

The following (somewhat surprising) theorem may be regarded as a converse of the previous proposition:

Theorem 83 (Sobolev Embedding Theorem)**.** *For $s > k + n/2$ we find constants $C = C(n, k, s)$ such that*

$$\|u\|_{C^k} \le C\|u\|_{W^s}$$

for all $u \in C^\infty(T^n)$. Passing to the completions, the identity map therefore induces a continuous embedding

$$W^s(T^n) \to C^k(T^n).$$

Proof. Using the Fourier expansion of $u \in C^\infty(T^n)$ and the triangle inequality for $\|\cdot\|_{C^k}$ we find

$$\|u\|_{C^k}^2 = \left\| \sum_{\nu\in\mathbb{Z}^n} \hat{u}(\nu) u_\nu \right\|_{C^k}^2 \leq \left(\sum_{\nu\in\mathbb{Z}^n} |\hat{u}(\nu)| \cdot \|u_\nu\|_{C^k} \right)^2$$

(we will show that the right hand side is finite, so that the series $\sum_{\nu\in\mathbb{Z}^n} \hat{u}(\nu)u_\nu$ converges absolutely in the Banach space C^k.)

From $\frac{\partial u_\nu}{\partial x^\alpha} = (i\nu)^\alpha u_\nu$ we find $\|u_\nu\|_{C^k}^2 = \max_{|\alpha|\leq k} \|\nu^\alpha\|^2 \|u_\nu\|_\infty^2 \leq (2\pi)^{-n}(1+\|\nu\|^2)^k$, so that

$$\|u\|_{C^k}^2 \leq (2\pi)^{-n} \left(\sum_{\nu\in\mathbb{Z}^n} |\hat{u}(\nu)| \cdot (1+\|\nu\|^2)^{k/2} \right)^2 = (2\pi)^{-n} \left(\sum_{\nu\in\mathbb{Z}^n} |\hat{u}(\nu)| \cdot (1+\|\nu\|^2)^{s/2} \cdot (1+\|\nu\|^2)^{(k-s)/2} \right)^2$$

Now an application of the Cauchy-Schwarz Inequality in ℓ^2 gives

$$\|u\|_{C^k}^2 \leq (2\pi)^{-n} \left(\sum_{\nu\in\mathbb{Z}^n} |\hat{u}(\nu)|^2 (1+\|\nu\|^2)^s \right) \left(\sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|^2)^{k-s} \right) \leq C \cdot \|u\|_{W^s}^2$$

for the constant $C = (2\pi)^{-n} \int_{\mathbb{R}^n} (1+|x|^2)^{k-s} dx$, which converges precisely when $k-s < -n/2$. This proves the inequality stated in the theorem, so that the identity map on $C^\infty(T^n)$ may be extended to a continuous map $F\colon W^s \to C^k$. This map is injective, using the same argument as above (C^k-convergence implies L^2-convergence on the torus). $\square$

Remark 84. *The definition of the norm $\|\cdot\|_{W^s}$ clearly also makes sense for real $s > 0$. The theorems in this section continue to hold for these more general Sobolev spaces.*

Theorem 85 (Rellich's Theorem)**.** *For $k_1 < k_2$ the inclusion $W^{k_2}(T^n) \to W^{k_1}(T^n)$ is compact.*

This means that any $\|\cdot\|_{W^{k_2}}$ bounded sequence has a convergent subsequence for the norm $\|\cdot\|_{W^{k_1}}$.

Proof. Let $B = \{u \in W^{k_2} \mid \|u\|_{W^{k_2}} \leq 1\}$ denote the unit ball in W^{k_2}. For $N \in \mathbb{N}$ define $Z_N = \{u \in W^{k_2} \mid \hat{u}(\nu) = 0 \ \forall |\nu| < N\}$. For $u \in Z_N$ we have the estimate

$$\|u\|_{W^{k_1}}^2 = \sum_{|\nu|\geq N} |\hat{u}(\nu)|^2 (1+\|\nu\|^2)^{k_2} (1+\|\nu\|^2)^{k_1-k_2} \leq (1+N^2)^{k_1-k_2} \cdot \|u\|_{W^{k_2}}^2. \tag{19}$$

Let $(u_n) \in B$. As W^{k_2}/Z_N is finite-dimensional we successively find subsequences for which $\hat{u}_{n_k^{(N)}}(\nu)$ converges for all $|\nu| < N$. Passing to the diagonal gives a subsequence $v_k = u_{n_k^{(k)}}$ for which all $\hat{v}_n(\nu)$ converge. It remains to show that (v_n) is a Cauchy sequence in $\|\cdot\|_{W^{k_1}}$.

Let $\varepsilon > 0$. Pick N with $(1+N^2)^{k_1-k_2} < \varepsilon^2$. Then by (19)

$$\begin{aligned}\|v_n - v_m\|_{W^{k_1}} &\leq \underbrace{\Big\|v_n - \sum_{|\nu|<N} \hat{v}_n(\nu)u_\nu\Big\|}_{\in Z_N\cap B}{}_{W^{k_1}} + \underbrace{\Big\|\sum_{|\nu|<N} \hat{v}_m(\nu)u_\nu - v_m\Big\|}_{\in Z_N\cap B}{}_{W^{k_1}} + \sum_{|\nu|<N} \|(\hat{v}_n(\nu) - \hat{v}_m(\nu))u_\nu\|_{W^{k_1}} \\ &\leq \varepsilon + \varepsilon + \sum_{|\nu|<N} |\hat{v}_n(\nu) - \hat{v}_m(\nu)|(1+\|\nu\|^2)^k \leq 2\varepsilon + (1+N^2)^k \sum_{|\nu|<N} |\hat{v}_n(\nu) - \hat{v}_m(\nu)|\end{aligned}$$

For n, m sufficiently large, the last summand is also $< \varepsilon$. $\square$

We now define Sobolev spaces of sections of vector bundles over closed Riemannian manifolds. Let $E \to M$ be a vector bundle with inner product and connection (not necessarily compatible with the inner product), defined over a closed Riemannian manifold (M, g). Given a section $u \in C^\infty(E)$ the covariant derivative is a section $\nabla u \in C^\infty(T^*M \otimes E)$. On the tensor product $T^*M \otimes E$ we use the connection characterized for $\alpha \in C^\infty(T^*M), s \in C^\infty(E)$ by

$$\nabla_X^{T^*M\otimes E}(\alpha \otimes s) = \nabla_X^{\mathrm{LC}}(\alpha) \otimes s + \alpha \otimes \nabla_X^E(s).$$

Then $\nabla\nabla u \in C^\infty(T^*M \otimes T^*M \otimes E)$, and so forth. In the case of $E = M \times \mathbb{C}$, $M = T^n$ a section may be viewed as a function $u\colon T^n \to \mathbb{C}$ and we have

$$\begin{aligned} \nabla(u) &= du = \partial_i u dx^i \\ \nabla\nabla(u) &= \partial_j\partial_i u dx^i \otimes dx^j \\ \nabla\cdots\nabla(u) &= \frac{\partial^{|\alpha|}u}{\partial x^\alpha} dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_k} \end{aligned}$$

which is the usual higher total differential.

Definition 86. *Let M be closed Riemannian manifold and let $E \to M$ be a vector bundle with inner product and connection ∇. For $k \in \mathbb{N}$ the k-*th Sobolev norm *of a section $u \in C^\infty(E)$ is defined as*

$$\|u\|_{W^k}^2 = \sum_{a=0}^{k} \left\|\nabla^a u\right\|_{L^2}^2, \qquad \nabla^a u = \underbrace{\nabla\cdots\nabla}_{a\ times} u.$$

*The Hilbert space completion of $C^\infty(E)$ for this norm is the k-*th Sobolev space of sections *$W^k(E)$.*

From (18) and the calculations preceding the definition it follows that this new Sobolev norm is equivalent to the old one (in the case $M = T^n$ and the trivial line bundle E). Definition 86 is more intuitive than Definition 78, however, the Sobolev and Rellich theorems rely on Fourier decompositions, which is most conveniently carried out on the torus.

Definition 87. *For $k \in \mathbb{N}$ we define similarly as in Definition* 81 *above*

$$\|u\|_{C^k} = \max_{0\le a\le k} \|\nabla^a u\|_\infty$$

(*note that this norm is not induced by an inner product.*) *The completion of the space $C^\infty(E)$ with respect to this norm is the Banach space $C^k(E)$.*

Proposition 88. *Over a closed manifold M, the equivalence class of the so-defined norm $\|\cdot\|_{W^k}$ is independent of the choice of metrics on M, E and the choice of connection ∇ on E. In particular, on T^n it is equivalent to the norm defined by* (18)*.*

For the proof one selects a finite cover of M by coordinate balls $U_i \approx B_1(0)$. The bundle E can be isometrically trivialized over each U_i (because it is contractible) and we may restrict attention to sections that are compactly supported in U_i. This is because, using a partition of unity, any section may be written as a finite sum of sections with support in U_i. Then the metric g and connection ∇ are determined on U_i by the components g_{ij} of the Riemannian metric tensor and the Christoffel symbols Γ_{pq}^r. All these (real or complex valued) functions and their derivatives are bounded on the relatively compact set $U_i \subset M$ and the bounds can be estimated against each other (from above and from below) for any two choices of g and ∇ on M.

For example, for functions on $\mathbb{R}^n$ with compact support within the relatively compact unit ball $B_1(0) \subset \mathbb{R}^n$, one can directly check that (for a non-standard metric g and connection ∇) that the Sobolev norm

$$\|u\|_{W^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$$

is equivalent to $\|u\|_{L^2}^2 + \sum_{i=1}^n \|\partial_{x^i} u\|_{L^2}^2$ where we use the standard derivative of functions on $\mathbb{R}^n$ and the standard volume element on $\mathbb{R}^n$. More details can be worked out in Exercise 1 on sheet 8.

Similarly, one shows that the norm $\|\cdot\|_{W^k}$ on $C^\infty(M)$ is equivalent to the following one, defined in terms of local trivializations: the sets U_i are diffeomorphic to open subsets of T^n. Let χ_i^2 be a partition of unity for this cover. Then $\|\cdot\|_{W^k}$ is equivalent to

$$\|u\|^2 = \sum_i \|u_i\|^2_{W^k(T^n)} \tag{20}$$

where $u_i = \chi_i \cdot u$ are regarded as $\mathbb{C}^{\mathrm{rk}(E)}$-valued functions on T^n (using isometric trivializations of E and charts).

Similar arguments show that the C^k-norm (87) is does not depend - up to equivalence - on the chosen metrics on M and E and the choice of connection on E.

Theorem 89. *Let $E \to M$ be a bundle with inner product and connection on a closed Riemannian manifold M. Then the identity map $C^\infty(E) \to C^\infty(E)$ induces*

1. *bounded inclusions $C^k(E) \to W^k(E)$,*
2. [Sobolev Embedding Theorem] *bounded inclusions $W^s(E) \hookrightarrow C^k(E)$ for all $s > k + n/2$,*
3. [Rellich] *compact inclusions $W^{k_2}(E) \hookrightarrow W^{k_1}(E)$ for all $k_2 > k_1$.*

Proof. Choose $U_i \approx B_1(0) \subset (-\pi,\pi)^n \subset T^n$ as above. Writing $u = \sum \chi_i u$, we see that is suffices to work in the space $C^\infty_{\mathrm{supp}\subset U_i}(E)$ of smooth functions with support in U_i, where the different norms are equivalent to our previously considered norms on T^n, compare Equation (20).

□

Proposition 90. *Let $E, F \to M$ be vector bundles with metrics and connections over a closed Riemannian manifold (M,g). Let $P\colon C^\infty(E) \to C^\infty(F)$ be a differential operator of order $\le k$. Then P extends to bounded linear maps $W^{k+l}(E) \to W^l(F)$ and $C^{k+l}(E) \to C^l(F)$.*

Proof. Being local, the operator P takes $C_c^\infty(U_i, E)$ to $C_c^\infty(U_i, F)$. Let $u \in C^\infty(M)$ and $u = \chi_i \cdot u$ as above. In trivializations over the chart neighborhood U_i we may write

$$Pu_i = \sum_{|\alpha|\le k} A^\alpha(x) \frac{\partial^{|\alpha|} u_i}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}.$$

The operator norms of the matrices $A^\alpha(x)$ (and their derivatives) are bounded on U_i. Using the Leibniz rule for higher derivatives, we get

$$\begin{aligned}\|Pu_i\|^2_{W^l} &\le C \sum_{|\beta|\le l} \Big\|\frac{\partial^{|\beta|}}{\partial x^\beta}\Big(\sum_{|\alpha|\le k} A^\alpha \frac{\partial^{|\alpha|} u_i}{\partial x^\alpha}\Big)\Big\|^2_{L^2} \\ &\le C \sum_{|\gamma|+|\delta|\le l} \sum_{|\alpha|\le k} \Big\|\frac{\partial^{|\gamma|} A^\alpha}{\partial x^\gamma}\Big\|^2_\infty \cdot \Big\|\frac{\partial^{|\alpha|+|\delta|} u_i}{\partial x^{\alpha+\delta}}\Big\|^2_{L^2} \le C\|u_i\|^2_{W^{l+k}} \le C\|u\|^2_{W^{l+k}}\end{aligned}$$

for a generic constant C. Summing up over i, the result follows from (20). The argument for the C^k-norms is similar. □

4.3 Analysis of Dirac Operators

Let (M,g) be a closed Riemannian manifold and let $S \to M$ be a Dirac bundle with corresponding Dirac operator $D\colon C^\infty(S) \to C^\infty(S)$. Obviously $D \in \mathcal{D}_1(S,S)$ is of first order. By Proposition 90 we have

$$\|Ds\|_{L^2} \le C\|s\|_{W^1}$$

for some $C > 0$. The Gårding inequality is a non-trivial converse of this inequality:

Theorem 91 (Gårding inequality)**.** *There is $C > 0$ so that $\|s\|_{W^1} \leq C(\|s\|_{L^2} + \|Ds\|_{L^2})$ for all $s \in C^\infty(S)$.*

Proof. Recall that $D^2(s) = \nabla^*\nabla(s) + K(s)$ by the Weitzenböck formula (Theorem 33). Therefore

$$\langle Ds, Ds\rangle_{L^2} = \langle D^2 s, s\rangle = \langle \nabla^*\nabla s, s\rangle + \langle Ks, s\rangle = \langle \nabla s, \nabla s\rangle + \langle Ks, s\rangle$$

and so $\|Ds\|_{L^2}^2 = \|\nabla s\|_{L^2}^2 + \langle Ks, s\rangle$. On a coordinate patch $U \subset M$ we have

$$\begin{aligned} \|\nabla s\|_{L^2}^2 &= \int_U g^{ij}(\partial_i s, \partial_j s) + 2g^{ij}\mathrm{Re}(\partial_i s, \Gamma_j s) + g^{ij}(\Gamma_i s, \Gamma_j s) \\ &\geq C_1\|s\|_{W^1}^2 - C_2\|s\|_{W^0}\|s\|_{W^1} \end{aligned}$$

using that $\nabla_i = \partial_i + \Gamma_i$. Hence, summing over finitely many coordinate patches covering M,

$$\|Ds\|_{W^0}^2 \geq C_3\|s\|_{W^1}^2 - C_4\|s\|_{W^0}\cdot\|s\|_{W^1}\,.$$

For every ε there is K so that $ab \leq \varepsilon a^2 + Kb^2$ (for all $a, b > 0$). This is clear because for $x = a/b$ the function $x - \varepsilon x^2$ is bounded above by some constant K. Using $\varepsilon = \frac{1}{2}\frac{C_3}{C_4}$ we find

$$C_4\|s\|_{W^1}\cdot\|s\|_{W^0} \leq \frac{1}{2}C_3\|s\|_{W^1}^2 + K\|s\|_{W^0}^2$$

so that

$$\|Ds\|_{W^0}^2 \geq \frac{C_3}{2}\|s\|_{W^1}^2 - K\|s\|_{W^0}^2$$

from which the Gårding inequality easily follows (note that $\|\cdot\|_{W^0} = \|\cdot\|_{L^2}$ by definition). $\square$

Theorem 92 (Elliptic Estimates)**.** *Let S be a Dirac bundle over a closed Riemannian manifold M. For $k \in \mathbb{N}$ there are constants $C_k > 0$ with*

$$\|s\|_{W^{k+1}} \leq C_k(\|s\|_{W^k} + \|Ds\|_{W^k}), \qquad s \in C^\infty(S).$$

for all $s \in W^{k+1}(S)$.

Proof. We proceed by induction, the case $k = 0$ being the Gårding inequality. For the induction step we can assume that $s \in C^\infty(S)$ by an approximation argument. In local coordinates

$$\|s\|_{W^{k+1}} \leq A_1\sum_{i=1}^n \|\partial_i s\|_{W^k} \leq A_1 \sum C_{k-1}\sum_{i=1}^n(\underbrace{\|\partial_i s\|_{W^{k-1}}}_{\leq A_2\|s\|_{W^k}} + \|D\partial_i s\|_{W^{k-1}})$$

by induction. Moreover, for the second term we have

$$\|D\partial_i s\|_{W^{k-1}} \leq \|\partial_i Ds\|_{W^{k-1}} + \|[D, \partial_i]s\|_{W^{k-1}} \leq A_3\|Ds\|_{W^k} + A_4\|s\|_{W^k}$$

where we use Proposition 90 applied to D and the differential operator $[D, \partial_i] \in \mathcal{D}_1$ of first order (for example, $[f\partial_1, \partial_2] = (\partial_2 f)\partial_1$ and similarly in the case of the Dirac operator). $\square$

Interlude: Unbounded Operators

Let $\mathcal{H}$ be a separable Hilbert space (such as $\ell^2(\mathbb{N})$ or $W^k(E)$). The Dirac operator is a linear map

$$D\colon C^\infty(S) \to C^\infty(S)$$

on the dense subspace $C^\infty(S)$ of the Hilbert space $L^2(S)$. We have an estimate $\|Ds\|_{L^2} \leq C\|s\|_{W^1}$, but $\|Ds\|_{L^2}$ cannot be controlled by $\|s\|_{L^2}$.

Definition 93. *An* unbounded operator *on a Hilbert space $\mathcal{H}$ is a linear map $A\colon \mathrm{Dom}(A) \to \mathcal{H}$ defined on a dense vector subspace* $\mathrm{Dom}(A) \subset \mathcal{H}$.

Recall that by the *Closed Graph Theorem* a linear map $A\colon \mathcal{H} \to \mathcal{H}$ is bounded precisely when its graph $\Gamma(A)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$.

Definition 94. *A unbounded operator $A\colon \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$ is said to be* closable *if $\overline{\Gamma(A)} \subset \mathcal{H} \times \mathcal{H}$ is the graph of a (uniquely determined and linear) map $\bar{A}\colon \mathcal{H} \supset \mathrm{Dom}(\bar{A}) \to \mathcal{H}$. Equivalently, $(0, y) \in \overline{\Gamma(A)}$ implies $y = 0$.*

Proposition 95. *Let $E \to M$ be a Hermitian vector bundle over a closed Riemannian manifold. A differential operator $P \in \mathcal{D}_k(E)$ defines an unbounded operator $L^2(E) \supset \mathrm{Dom}(P) = C^\infty(E) \to L^2(E)$. The operator P is closable.*

Proof. Suppose $x_i \to 0$ and $Px_i \to y$ in L^2, where $x_i \in C^\infty(E)$. We must show $y = 0$. Consider the inner product with an arbitrary $x \in C^\infty(E)$:

$$\langle x, y\rangle_{L^2} = \lim \langle x, Px_i\rangle_{L^2} = \lim \langle P^* x, x_i\rangle_{L^2} = \langle P^* x, \lim x_i\rangle_{L^2} = 0$$

Since $C^\infty(E)$ is a dense subspace of $L^2(E)$, it follows that $y = 0$, as required. □

Proposition 96. *Let $S \to M$ be a Dirac bundle. Then* $\mathrm{Dom}(\bar{D}) = W^1(S)$.

Proof. $s \in \mathrm{Dom}(\bar{D})$ is equivalent to the existence of a L^2-convergent sequence $s_i \to s$, $s_i \in C^\infty(S)$, with Ds_i convergent in L^2. Then the sequence s_i is W^1-Cauchy, by the Gårding inequality:

$$\|s_i - s_j\|_{W^1} \leq C(\|s_i - s_j\|_{L^2} + \|Ds_i - Ds_j\|_{L^2}).$$

It follows that the limit s of the W^1-convergent sequence s_i also belongs to W^1. This proves $\mathrm{Dom}(\bar{D}) \subset W^1(S)$. Conversely, if $s \in W^1(S)$ then we have a W^1-convergent sequence $s_i \to s$. Then also $s_i \to s$ in L^2 and $Ds_i \to Ds$ in L^2 by Proposition 90. □

Note that by Proposition 90 the closure $\bar{D}\colon W^1(S) \to L^2(S)$ is bounded.

Definition 97. *Let $P \in \mathcal{D}_k(E)$ be a differential operator. Let $x, y \in L^2(E)$. We say that $Px = y$* weakly *if*

$$\langle x, P^*\varphi\rangle = \langle y, \varphi\rangle \qquad \forall \varphi \in C^\infty(E).$$

If x, y are smooth and $Px = y$, then by definition of the adjoint differential operator, $Px = y$ weakly.

Question: If $Px = 0$ weakly, does it follow that $Px = 0$ (meaning that $x \in C^\infty(E)$ and that x lies in the kernel of P)?

The point of weak solutions is that we may use abstract Hilbert space theory to construct them. The affirmative answer to our question later for $P = D$ will then give us actual solutions of the PDE $Dx = 0$.

Smoothing Operators

Definition 98. *Let $E, F \to M$ be vector bundles with inner product over a closed Riemannian manifold. An operator $A\colon C^\infty(E) \to C^\infty(F)$ given by the formula*

$$(Au)(y) = \int_M K(y, x)u(x)\mathrm{dvol}(x)$$

where $K \in C^\infty(F \boxtimes E^*)$ (the smoothing kernel*) is called a* smoothing operator.

Here $E \boxtimes F = \mathrm{pr}_1^* E \otimes \mathrm{pr}_2^* F$ denotes the exterior tensor product of vector bundles $E \to X$ and $F \to Y$. This is a bundle over $X \times Y$ with fiber over the point (x, y) given by $E_x \otimes F_y$.

Proposition 99. *1. The operator A admits a unique extension $A\colon L^2(E) \to L^2(E)$.*

2. We have $A(L^2(E)) \subset C^\infty(F)$.

3. For all $k \geq 0$ the operator $A\colon L^2(E) \to C^\infty(F) \subset W^k(F)$ is continuous (for the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^k}$).

(the last point motivates calling A a differential operator of order $-\infty$)

Proof. For $u \in C^\infty(E)$ we have, using the Cauchy-Schwarz inequality

$$\begin{aligned}
\|Au\|_{L^2}^2 &= \int_M \|Au(y)\|^2 \mathrm{dvol}(y) = \int_M \left\| \int_M K(y,x)u(x)\mathrm{dvol}(x) \right\|^2 \mathrm{dvol}(y) \\
&= \int_M \left(\int_M \|K(y,x\|^2 \mathrm{dvol}(x) \cdot \int_M \|u(x)\|^2 \mathrm{dvol}(x) \right) \mathrm{dvol}(y) \\
&= \left(\int_M \int_M \|K(y,x\|^2 \mathrm{dvol}(x)\mathrm{dvol}(y) \right) \cdot \int_M \|u(x)\|^2 \mathrm{dvol}(x) \\
&= \|K\|_{L^2}^2 \cdot \|u\|_{L^2}^2
\end{aligned}$$

This shows that $\|A\| \leq \|K\|_{L^2}$ for the operator norm (in fact, they are equal), which proves 1. Item 2. is immediate by differentiating under the integral sign. For 3., consider $P \in \mathcal{D}_k(F)$. Then $P \circ A$ is again a smoothing operator with kernel $P\left(K(-,x)\right)$. Applying 1. to $P \circ A$ for $P = \nabla^i$, $0 \leq i \leq k$ we get

$$\|\nabla^i \circ A(u)\|_{L^2} \leq C_i \|u\|_{L^2}.$$

This gives an inequality $\|Au\|_{W^k} \leq D_k \|u\|_{L^2}$. ☐

In order to approximate L^2-sections by smooth sections we will need families of smoothing operators:

Definition 100. *A family $F_\varepsilon\colon L^2(E) \to L^2(E)$ for $\varepsilon \in (0,1]$ of smoothing operators is called a* Friedrichs mollifier *if*

- *Every F_ε is self-adjoint, meaning $\langle F_\varepsilon x, y\rangle = \langle x, F_\varepsilon y\rangle$ for all $x, y \in L^2(E)$.*
- *The family F_ε is uniformly bounded, meaning that we find $C > 0$ with $\|F_\varepsilon\| \leq C$ $(\forall \varepsilon \in (0,1])$.*
- *If $B \in \mathcal{D}_1(E)$ then $[B, F_\varepsilon]\colon C^\infty(E) \to C^\infty(E)$ induces a family of bounded operators $L^2(E) \to L^2(E)$ with uniform bound C, meaning $\|[B, F_\varepsilon]\| \leq C$ $(\forall \varepsilon \in (0,1])$.*
- *We have $F_\varepsilon \to \mathrm{id}_{L^2(E)}$ in the weak operator topology, meaning that $\langle F_\varepsilon x, y\rangle \to \langle x, y\rangle$ for all $x, y \in L^2$.*

On exercise sheet 8 we shall see that Friedrichs mollifiers do indeed exist.

Definition 101. *Let $A\colon \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$ be an unbounded operator. The adjoint A^* is the unbounded operator $A^*\colon \mathcal{H} \supset \mathrm{Dom}(A^*) \to \mathcal{H}$ where*

$$\mathrm{Dom}(A^*) = \{y \in \mathcal{H} \mid \mathrm{Dom}(A) \to \mathcal{H}, x \mapsto \langle Ax, y\rangle \text{ bounded}\}$$

For $y \in \mathrm{Dom}(A^*)$ the functional $\langle A-, y\rangle$ may be extended to $\mathcal{H}$, so the Riesz Representation Theorem asserts the existence of a unique $z \in \mathcal{H}$ with $\langle A-, y\rangle = \langle -, z\rangle$. We define $A^* y = z$. Thus

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \quad \forall x \in \mathrm{Dom}(A), y \in \mathrm{Dom}(A^*).$$

Note that $\mathrm{Dom}(A^*)$ needn't be a dense subspace of $\mathcal{H}$, so strictly speaking A^* isn't necessarily an unbounded operator.

Definition 102. *An unbounded operator $A\colon \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$ is called* symmetric *if*

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y \in \mathrm{Dom}(A).$$

In this case, $\mathrm{Dom}(A) \subset \mathrm{Dom}(A^*)$.

Definition 103. *An unbounded operator* $A\colon \mathcal{H} \supset \mathrm{Dom}(A) \to \mathcal{H}$ *is* essentially self-adjoint *in case*

1. *A is closable,*
2. *$\bar{A}$ is self-adjoint, meaning* $\mathrm{Dom}(\bar{A}^*) = \mathrm{Dom}(\bar{A})$ *and* $\bar{A} = \bar{A}^*$.

Example 104. *For a Dirac bundle $S \to M$ over a closed Riemannian manifold the Dirac operator D is essentially self-adjoint. By definition, $y \in \mathrm{Dom}(\bar{D}^*)$ means that we find $z \in L^2(S)$ with $\langle \bar{D}x, y\rangle = \langle x, z\rangle$ for all $x \in W^1(S)$. This is equivalent to*

$$\langle Ds, y\rangle = \langle s, z\rangle \quad \forall s \in C^\infty(S),$$

i.e. that $Dy = z$ weakly. *Our assertion now follows from Proposition* 107 *below.*

Before undertaking the proof, we review the notion of weak convergence.

Definition 105. *Let (u_n) be a sequence in a Hilbert space $\mathcal{H}$. We say that $u_n \rightharpoonup u$* weakly *if*

$$\langle u_n, s\rangle \to \langle u, s\rangle \quad \forall s \in \mathcal{H}$$

Remark 106.

1. *Since $\langle , \rangle$ is positive definite, such a* weak limit *s is unique.*
2. *If $u_n \to u$, then $u_n \rightharpoonup u$. The converse is false (for example, take $\mathcal{H} = \ell^2(\mathbb{N})$ and the standard basis $u_n = e_n$).*
3. *If $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator and $u_n \rightharpoonup u$, then $Au_n \rightharpoonup Au$.*
4. *Every bounded sequence (u_n) in $\mathcal{H}$ possesses a weakly convergent subsequence (Theorem of Banach-Alaoglu).*

Proposition 107. *Let $y, z \in L^2(S)$ and $Dy = z$ weakly. Then $y \in W^1(S) = \mathrm{Dom}(\bar{D})$ and $\bar{D}y = z$.*

Proof. Let (F_ε) be a Friedrichs mollifier for $S \to M$. Define $y_\varepsilon = F_\varepsilon(y)$. According to Definition 100, we have $y_\varepsilon \rightharpoonup y$ weakly. For $s \in C^\infty(S)$ we have

$$|\langle Dy_\varepsilon, s\rangle| = |\langle DF_\varepsilon y, s\rangle| = |\langle y, F_\varepsilon Ds\rangle| \leq \underbrace{|\langle y, DF_\varepsilon s\rangle|}_{=\langle z, F_\varepsilon s\rangle \leq C_1\|s\|} + \underbrace{|\langle y, [D, F_\varepsilon]s\rangle|}_{\leq C_2\|s\|}$$

The first equality in the underbrace uses the fact that $Dy = z$ weakly and the inequality in the second underbrace uses the fact that $[D, F_\epsilon]$ is globally bounded by the properties of Friedrichs mollifiers. It follows that $\|Dy_\varepsilon\|$ is bounded in the L^2-norm. Combining this with Gårding's inequality, we see that $\|y_\varepsilon\|_{W^1}$ is bounded, so we obtain a weakly convergent subsequence $y_\varepsilon \rightharpoonup y'$ for some $y' \in W^1(S)$. Since $W^1 \to L^2$ is bounded, it follows that $y_\varepsilon \rightharpoonup y'$ in L^2. The uniqueness of weak limits implies $y' = y$. This proves $y \in W^1$.

Let $\bar{D}y = z' \in L^2(S)$. By definition, we find an L^2-convergent sequence $y_n \to y$ with $y_n \in C^\infty(S)$ and $Dy_n \to z'$. We wish to prove $z = z'$. For $s \in C^\infty(S)$ we have

$$\langle z', s\rangle \leftarrow \langle Dy_n, s\rangle = \langle y_n, Ds\rangle \to \langle y, Ds\rangle = \langle z, s\rangle$$

It follows that $z = z'$. ∎

Theorem 108 (Elliptic Regularity)**.** *Let $s \in W^1(S)$ and $\overline{D}s = 0$. Then $s \in C^\infty(S)$ is smooth (and of course $Ds = 0$).*

Proof. As a preparation we show that for all k the operators

$$F_\varepsilon, [D, F_\varepsilon]\colon W^k(S) \to W^k(S),$$

are uniformly bounded in ε, compare Roe's proof of Proposition 5.24. (The assertion for $[D, F_\epsilon]$ is sometimes called Friedrichs Lemma). For estimating $[D, F_\epsilon]$ we need to work with special mollifiers defined by convolution, compare Roe's Exercise 5.34., respectively our Exercise 2 on Sheet 8.

For $k = 0$ the above claims follow from Definition 100.

For the inductive step, we first show that the W^{k+1}-norm of F_ϵ is uniformly bounded. Let $s \in W^{k+1}(S)$. We use Theorem 92, the induction hypothesis, and Proposition 90 to see

$$\begin{aligned}\|F_\varepsilon s\|_{W^{k+1}} &\le C\left(\|F_\varepsilon s\|_{W^k} + \|DF_\varepsilon s\|_{W^k}\right)\\ &\le C\left(\|F_\varepsilon s\|_{W^k} + \|F_\varepsilon Ds\|_{W^k} + \|[D, F_\varepsilon]s\|_{W^k}\right) \le D\|s\|_{W^{k+1}}.\end{aligned}$$

We show next that the W^{k+1}-norms of $[D, F_\varepsilon]$ are uniformly bounded. By a partition of unity we can work in a local chart neighborhood $\mathbb{R}^n$, where $S = \mathbb{R}^n \times \mathbb{R}^l$ is trivial. Let (F_ε) be Friedrichs mollifiers, defined by convolution with $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, where

$$\phi(x) = \begin{cases}\exp(-1/(1 - \|x\|^2)) & \|x\| < 1,\\ 0 & \|x\| \ge 1.\end{cases}$$

For a section $s\colon \mathbb{R}^n \to \mathbb{R}^l$ we define

$$F_\varepsilon s(x) = \int \phi_\varepsilon(x - y)s(y)dy.$$

Using integration by parts it is easy to check that F_ε commutes with all differential operators ∂_i. Compare Exercise 5.34. (iv) in Roe (for differential operators B with constant coefficients).

By definition of the Sobolev norms, it suffices to prove that $\partial_i[D, F_\varepsilon]$ defines a uniformly bounded family of operators $W^{k+1} \to W^k$ for $i = 1, \dots, n$. For this we use local coordinates to write $D = D^j\partial_j$ with $D^j \in C^\infty(\mathbb{R}^n, \mathbb{R}^l)$ bounded. Then, using that F_ε commutes with all ∂_i we have

$$\begin{aligned}&\partial_i[D, F_\varepsilon] = (\partial_i D^j)F_\varepsilon\partial_j + D^jF_\varepsilon\partial_{ij} - F_\varepsilon(\partial_i D^j)\partial_j - F_\varepsilon D^j\partial_{ij}\\ &[D, F_\varepsilon]\partial_i = D^jF_\varepsilon\partial_{ij} - F_\varepsilon D^j\partial_{ij}\end{aligned}$$

Hence

$$\partial_i[D, F_\varepsilon] = [D, F_\varepsilon]\partial_i + (\partial_i D^j)F_\varepsilon\partial_j - F_\varepsilon(\partial_i D^j)\partial_j$$

All three families $W^{k+1} \xrightarrow{\partial_i} W^k \xrightarrow{[D,F_\varepsilon]} W^k$, $W^{k+1} \xrightarrow{\partial_j} W^k \xrightarrow{F_\varepsilon} W^k \xrightarrow{\mathrm{mult}_{\partial_i D^j}} W^k$, and $W^{k+1} \xrightarrow{\partial_j} W^k \xrightarrow{\mathrm{mult}_{\partial_i D^j}} W^k \xrightarrow{F_\varepsilon} W^k$, are uniformly bounded by induction. Here we note that the multiplication operator with the bounded function $\partial_i D^j$ defines a bounded map $W^k \to W^k$.

After this preparation, we may now prove the claim of the theorem: $s \in W^k(S)$ for all k. We work by induction. By assumption $s \in W^1$. Let's assume $s \in W^k$. Since $Ds = 0$ we have

$$\|F_\varepsilon s\|_{W^{k+1}} \le C_k\left(\|F_\varepsilon s\|_{W^k} + \underbrace{\|DF_\varepsilon s\|_{W^k}}_{=\|[D,F_\varepsilon]s\|_{W^k} \le C}\right)$$

Here we use the estimates for F_ϵ and $[D, F_\epsilon]$ proven before. Since $\|F_\varepsilon s\|_{W^{k+1}}$ is bounded, we find a weakly convergent subsequence $F_{\varepsilon_i}s \rightharpoonup \tilde{s}$ in W^{k+1}. But since $F_\varepsilon s \rightharpoonup s$ we get $s = \tilde{s} \in W^{k+1}(S)$. Because $s \in W^k(S)$ for all k, Sobolev's Theorem 83 implies $s \in C^\infty(S)$. $\square$

Eigenspace Decomposition of D

Recall that the Dirac operator is a self-adjoint operator $\bar{D}\colon L^2(S) \supset \mathrm{Dom}(\bar{D}) \to L^2(S)$. We denote the graph $\mathcal{G} = \Gamma(\bar{D}) = \overline{\Gamma(D)} \subset \mathcal{H} \oplus \mathcal{H}$. By Propositions 96 and 107 we know that

$$\mathcal{G} = \{(x,y) \in L^2(S) \times L^2(S) \mid Dx = y \text{ weakly}\} = \{(x,y) \in W^1(S) \times L^2(S) \mid \bar{D}x = y\}.$$

Lemma 109. *Let $J\colon \mathcal{H}\oplus\mathcal{H} \to \mathcal{H}\oplus\mathcal{H}, (x,y) \mapsto (-y,x)$. Then $\mathcal{H}\oplus\mathcal{H} = \mathcal{G} \oplus J\mathcal{G}$ is an orthogonal direct sum.*

Proof. $(x,y) \in \mathcal{G}^\perp$ means that for all $s \in C^\infty(S)$ we have $0 = \langle (x,y),(s,Ds)\rangle = \langle x,s\rangle + \langle y, Ds\rangle$. Equivalently, $x = -Dy$ weakly which is equivalent to $(-y,x) \in \mathcal{G}$ or to $(x,y) \in J\mathcal{G}$. $\square$

Definition 110. *Let $\mathrm{pr}_{\mathcal{G}}\colon \mathcal{H}\oplus\mathcal{H} \to \mathcal{G}$ denote the orthogonal projection onto the graph. Define $Q\colon L^2(S) \to L^2(S)$ by the equation $\mathrm{pr}_{\mathcal{G}}(x,0) = (Qx, \bar{D}Qx)$. In other words, we set Qx equal to y if $y, \bar{D}y \in W^1$ and $x = y + \bar{D}^2 y$ (by the lemma, a unique such y can be found for any $x \in L^2(S)$).*

We note the following properties:

1. $Qx \in W^1(S) = \mathrm{Dom}(\bar{D})$.
2. Since projections $\mathrm{pr}_{\mathcal{G}}$ have norm 1, we get $\|Qx\|_{L^2}^2 + \|\bar{D}Qx\|_{L^2}^2 \le \|x\|_{L^2}^2$. Thus $\|Qx\|_{L^2} \le \|x\|_{L^2}$ so for the operator norm of Q we get $\|Q\| \le 1$. Moreover, $\|\bar{D}Qx\|_{L^2} \le \|x\|_{L^2}$ and by Gårdings inequality, Q is a bounded operator $L^2(S) \to W^1(S)$:
$$\|Q(x)\|_{W^1} \le C\left(\|Qx\|_{L^2} + \|\bar{D}Qx\|_{L^2}\right) \le C\|x\|_{L^2}$$
Combined with Rellich's Theorem 85 we get a compact operator
$$Q\colon L^2(S) \to W^1(S) \to L^2(S)$$
3. Q is self-adjoint because $\mathrm{pr}_{\mathcal{G}}^* = \mathrm{pr}_{\mathcal{G}}$ (projection operators are self adjoint):
$$\langle Qx, y\rangle = \langle \underbrace{(Qx, \bar{D}Qx)}_{=\mathrm{pr}_{\mathcal{G}}(x,0)}, (y,0)\rangle = \langle (x,0), \underbrace{(Qy, \bar{D}Qy)}_{=\mathrm{pr}_{\mathcal{G}}(y,0)}\rangle = \langle x, Qy\rangle$$
4. Q is non-negative:
$$\langle Qx, x\rangle = \langle (Qx, \bar{D}Qx), (x,0)\rangle = \langle \mathrm{pr}_{\mathcal{G}}(x,0), (x,0)\rangle \ge 0$$
since projections are non-negative.
5. Q is injective: if $Qx = 0$ then $(x,0) \in \mathcal{G}^\perp = J\mathcal{G}$ so $x = -\bar{D}0 = 0$.

By the Spectral Theorem (see exercise sheet 9) for compact self-adjoint operators we get a sequence of real eigenvalues $1 \ge \alpha_1 > \alpha_2 > \cdots > 0$ tending to zero so that the entire Hilbert space may be decomposed into the corresponding finite-dimensional eigenspaces $\mathrm{Eig}(Q, \alpha_i)$ of the operator Q:

$$\mathcal{H} = \overline{\bigoplus_i \mathrm{Eig}(Q, \alpha_i)}, \qquad \mathrm{Eig}(Q,\alpha) = \{x \in L^2(S) \mid Qx = \alpha x\}.$$

Lemma 111. *For $0 < \alpha < 1$ let λ be the positive solution to $\lambda^2 = (1-\alpha)/\alpha$. Then*

$$\mathrm{Eig}(Q,\alpha) \subset \mathrm{Eig}(\bar{D},\lambda) \oplus \mathrm{Eig}(\bar{D},-\lambda). \tag{21}$$

Moreover for $\alpha = 1$ we have $\mathrm{Eig}(Q,1) \subset \ker(\bar{D}) = \mathrm{Eig}(\bar{D},0)$.

Proof. Let $x \in \mathrm{Eig}(Q, \alpha)$ for $0 < \alpha < 1$. Then $Qx \in W^1$ and by definition of Q we find $y \in W^1$ with

$$(\alpha x, \alpha \bar{D} x) + (-\bar{D} y, y) = (Qx, \bar{D} Qx) + (-\bar{D} y, y) = (x, 0) \in \mathcal{G} \oplus J\mathcal{G}.$$

This means $(\alpha - 1)x = \bar{D} y$, $y = -\alpha \bar{D} x$. Let $z = -\frac{1}{\alpha\lambda} y$. Then $\bar{D} x = \lambda z$, $\bar{D} z = \lambda x$ so

$$x + z \in \mathrm{Eig}(\bar{D}, \lambda), x - z \in \mathrm{Eig}(\bar{D}, -\lambda)$$

If $\alpha = 1$ then $\bar{D} y = 0, y = -\bar{D} x$ so $\bar{D}^2 x = 0$ and $\bar{D} x = 0$ (because $\|\bar{D} x\|^2 = \langle \bar{D} x, \bar{D} x\rangle = \langle \bar{D}^2 x, x\rangle = 0$). $\square$

Since the eigenspaces of Q are mutually orthogonal, we conclude equality in (21). In particular, all $\mathrm{Eig}(\bar{D}, \lambda)$ are finite dimensional. Summarizing we get

$$\mathcal{H} = \overline{\bigoplus_{\alpha_i} \mathrm{Eig}(\bar{Q}, \lambda)} = \overline{\ker(\bar{D}) \bigoplus_{\lambda_i > 0} \mathrm{Eig}(\bar{D}, \lambda_i) \bigoplus_{\lambda_j < 0} \mathrm{Eig}(\bar{D}, \lambda_j)}$$

for a discrete subset $\{\lambda_i, \lambda_j\} \subset \mathbb{R}$ with accumulation points only at $\pm\infty$. It follows from our argument and because $\mathcal{H}$ is infinite dimensional that $\lambda_i \to \infty$ or $\lambda_j \to -\infty$ or both.

We call $\sigma(D) := \{\lambda \in \mathbb{R} \mid \lambda \text{ eigenvalue of } \bar{D}\}$ the *spectrum* of D.

We get the following generalization of the Theorem 108.

Theorem 112 (Elliptic Regularity)**.** *Let $\lambda \in \sigma(D)$ and $s \in \mathrm{Eig}(\overline{D}, \lambda)$. Then $s \in C^\infty(S)$ is smooth. Hence all eigenspaces of $\overline{D}$ consist of smooth sections and are in fact eigenspaces of D.*

Proof. The proof of Gårding's inequality Theorem 91 and the elliptic estimates 92 apply as well to the operator $D_\lambda := D - \lambda \cdot \mathrm{id}$: Using the Weitzenböck formula we have $D_\lambda^2 = \nabla^* \nabla + K$ where K is a *first* order differential operator and this is in fact enough for the proof of Theorem 91 go through.

Hence the proof of Theorem 108 applies as well to $\overline{D_\lambda}$ instead of $\overline{D}$. $\square$

We hence get an orthogonal decomposition

$$L^2(S) = \overline{\bigoplus_{\lambda \in \sigma(D)} \mathrm{Eig}(D, \lambda)}$$

where each eigenspace is a finite dimensional subspace of $C^\infty(S)$ and the eigenvalues λ have accumulation points only at $\pm\infty$.

Remark 113.

- *One can show that the spectrum of D is in general not symmetric around* 0.
- *One can also show that the spectrum of D is neither bounded from below nor from above.*

Functional Calculus

Any $s \in L^2(S)$ may be decomposed orthogonally as

$$s = \sum_{\lambda \in \sigma(D)} s_\lambda, \quad \|s_\lambda\|_{L^2} \leq \|s\|_{L^2}, \;\; s_\lambda \in \mathrm{Eig}(D, \lambda) \tag{22}$$

where each s_λ is smooth.

As the next example shows, this may be regarded as a generalized Fourier expansion:

Example 114. *Let $D = -i\frac{d}{dt} \colon C^\infty(S^1, \mathbb{C}) \to C^\infty(S^1, \mathbb{C})$. Then $\sigma(D) = \mathbb{Z}$ and $\mathrm{Eig}(D, \lambda) = \mathbb{C}e^{i\lambda t}$.*

Proposition 115. *Let $s \in L^2(S)$. Then s is smooth if and only if for all k we have $|\lambda|^k \|s_\lambda\|_{L^2} \xrightarrow{|\lambda| \to \infty} 0$.*

Proof. We use the following estimate on the *eigenvalue growth* of D, proven further below in Proposition 128 [3]: For $\Lambda > 0$ let $N(\Lambda)$ be the number of eigenvalues of D (counted with multiplicity) whose norm is bounded by Λ.Then

$$N(\Lambda) \leq C(1+\Lambda)^{\frac{n(n+4)}{2}},$$

where $n = \dim M$ and C is a constant, which depends only on M and the rank of S.

Now for the proof of the proposition, note that $s_\lambda \in C^\infty$, by elliptic regularity. Applying Theorem 92 we have by induction

$$\|s_\lambda\|_{W^k} \leq C_k |\lambda|^k \|s_\lambda\|_{L^2}.$$

Now assume that $|\lambda|^\ell \, \|s_\lambda\|_{L^2} \xrightarrow{|\lambda|\to\infty} 0$ for all ℓ. Pick $k > 0$. Then, for all large enough $n \geq 0$, we have

$$\sum_{n<|\lambda|\leq n+1} \|s_\lambda\|_{W^k} \leq C_k \cdot N(n+1) \max_{n<|\lambda|\leq n+1} |\lambda|^k \|s_\lambda\|_{L^2} \leq \frac{1}{n^2}$$

because $n^2 \cdot C_k \cdot N(n+1) \max_{n<|\lambda|\leq n+1} |\lambda|^k \|s_\lambda\|_{L^2}$ tends to 0 as n goes to infinity by our assumption and the above estimate on $N(n+1)$. Hence $\sum_{\lambda\in\sigma(D)} s_\lambda$ converges absolutely in W^k-norm. We conclude $s \in W^k$ for all k and hence from Theorem 83, $s \in C^\infty$.

Conversely, if the series $s = \sum s_\lambda \in W^k(S)$ for all k, then $\sum |\lambda|^{2k} \|s_\lambda\|^2 = \|D^k s\|_{L^2}^2 < \infty$ for all k and hence $|\lambda|^k \, \|s_\lambda\|_{L^2} \xrightarrow{|\lambda|\to\infty} 0$ for all k. □

Example 116. *To deduce the convergence of the series $\sum_{k=0}^\infty \varphi(\lambda_k)$ for φ rapidly decreasing it is important to have growth estimates for (λ_k). Indeed, consider the slowly growing $\lambda_k = \ln(k)$ and $\varphi(\lambda) = \exp(-\lambda)$. Then φ is rapidly decreasing but $\varphi(\lambda_k) = 1/k$ which gives the non-convergent harmonic series.*

Definition 117 (Functional calculus)**.** *For a bounded function $f\colon \sigma(D) \to \mathbb{R}$ we define the operator*

$$f(D)\colon L^2(S) \to L^2(S), \quad \sum s_\lambda \mapsto \sum f(\lambda) s_\lambda$$

From the functional calculus we take the following facts:

1. $\|f(D)\| = \sup_{\lambda\in\sigma(D)} |f(\lambda)|$. In fact, $\sigma(f(D)) = f(\sigma(D))$.
2. The map $f \mapsto f(D)$ defines a ring homomorphism $\mathrm{Abb}_b(\mathbb{R}) \to B(L^2(S))$ on the ring $\mathrm{Abb}_b(\mathbb{R})$ of bounded functions $\mathbb{R} \to \mathbb{R}$ with point-wise addition and multiplication.
3. If $f \in O(|\lambda|^{-k})$ for all k (we call such f *rapidly decreasing*), then $f(D)(s) \in C^\infty(S)$ is smooth for any $s \in L^2(S)$. In fact, it is a smoothing operator (see Exercise sheet 9).
4. Let $f(x) = xg(x)$ for bounded f, g. Then $f(D) = \bar{D} \circ g(D) = g(D) \circ \bar{D}$ on $W^1(S)$.
5. $f(D)$ is self-adjoint.
6. For the constant function 1 we have $1(D) = \mathrm{id}_{L^2(S)}$.

Example 118. *Let $f_\varepsilon(\lambda) = \exp(-\varepsilon\lambda^2)$, $\varepsilon \geq 0$. If $\varepsilon > 0$ then $F_\varepsilon = f_\varepsilon(D)\colon L^2(S) \to C^\infty(S)$. For $s \in L^2(S)$ we have L^2-convergence $F_\varepsilon s \to s$, as can be seen by writing $s = \sum s_\lambda$ as an L^2-convergent series. Moreover F_ε is self-adjoint and $\|F_\varepsilon\|$ is uniformly bounded by 1. Since $\lambda f(\lambda) = f(\lambda)\lambda$ we have $F_\varepsilon \circ \bar{D} = \bar{D} \circ F_\varepsilon$ and hence $[\bar{D}, F_\epsilon] = 0$ on $W^1(S)$.*

[3] This consideration is errornously left out in [Roe] - compare Example 116

4.4 Application: Hodge Theory

Let (M, g) be an n-dimensional closed Riemannian manifold. Recall that the de Rham cohomology of M is

$$H^n_{dR}(M) = H^n(\Omega^*(M), d)$$

for the complex of differential forms $\Omega^k(M) = C^\infty(\Lambda^k T^*M)$. Thus any class $c \in H^k_{dR}(M)$ is represented by a closed k-form $\omega \in \Omega^k(M)$ (so $d\omega = 0$).

Motivating question: What is the 'best' representative ω for the class c?

The elements $c = [\omega]$ of $H^k_{dR}(M) = \ker(d^k)/\operatorname{im}(d^{k-1})$ may be viewed as affine subspaces $\omega + \operatorname{im}(d^{k-1})$ of $\ker(d^k)$. To single out a representative, we demand $\omega \perp \operatorname{im}(d^{k-1})$ using the L^2 inner product. This means

$$0 = \langle d\eta, \omega\rangle = \langle \eta, d^*\omega\rangle \qquad \forall \eta \in \Omega^{k-1}(M).$$

Our requirement is therefore equivalent to

$$(d^*\omega = 0 \text{ and } d\omega = 0) \Leftrightarrow D\omega = 0 \Leftrightarrow \Delta\omega = 0$$

for the Hodge-Dirac operator $D = d + d^*$. Recall that such a form ω is called *harmonic.* The vector space of harmonic k-forms will be denoted $\mathcal{H}^k(M)$.

Lemma 119. *Let $S \to M$ be a Dirac bundle. Then we have an orthogonal decomposition*

$$C^\infty(S) = \ker(D) \oplus \operatorname{im}\left(D\colon C^\infty(S) \to C^\infty(S)\right).$$

Proof. We know already that we have an orthogonal decomposition

$$L^2(S) = \ker(D) \oplus \overline{\bigoplus_{\lambda\in\sigma(D)\setminus\{0\}} \operatorname{Eig}(D, \lambda)}$$

where $\ker(D), \operatorname{Eig}(D, \lambda) \subset C^\infty(S)$. Write $\varphi = \varphi_0 + \tilde\varphi$ for $\varphi_0 \in \ker(D)$ and $\tilde\varphi \in \overline{\bigoplus_{\lambda\in\sigma(D)\setminus\{0\}} \operatorname{Eig}(D, \lambda)}$. Since φ, φ_0 are smooth, the section $\tilde\varphi$ is also smooth. We wish to find $\psi \in C^\infty(S)$ with $D\psi = \tilde\varphi$.

Let $\tilde\varphi = \sum \varphi_\lambda$ for $\varphi_\lambda \in \operatorname{Eig}(D, \lambda)$ with $\lambda \in \sigma(D) \setminus \{0\}$. Since $\tilde\varphi$ is smooth, the sequence $\|\varphi_\lambda\|_{L^2}$ is rapidly decreasing. Define $\psi = \sum_\lambda \frac{1}{\lambda}\varphi_\lambda$. Note that $\|\frac{1}{\lambda}\varphi_\lambda\|_{L^2} \in O(|\lambda|^{-k})$ for each k, so it is rapidly decreasing which shows that ψ is indeed smooth. Clearly also $D\psi = \tilde\varphi$. ☐

Corollary 120. *We have $\Omega^k(M) = \ker(D) \oplus \operatorname{im}(D)$ for the vector space space $\ker(D) = \mathcal{H}^k(M)$ of harmonic k-forms, which is finite-dimensional.*

As a consequence of elliptic regularity, we may now easily deduce:

Theorem 121 (Hodge Decomposition)**.** *We have an L^2-orthogonal decomposition*

$$\Omega^k(M) = \mathcal{H}^k \oplus \operatorname{im}(d^{k-1}) \oplus \operatorname{im}(d^{k+1})^*.$$

Moreover, $\ker(d) = \mathcal{H}^k \oplus \operatorname{im}(d^{k-1})$.

Proof. Since $D = d + d^*$, the image of $D\colon \Omega(M) \to \Omega(M)$ is $d\Omega(M) + d^*\Omega(M)$. Because of

$$\langle d\eta, d^*\omega\rangle = \langle dd\eta, \omega\rangle = 0$$

this decomposition is orthogonal and hence a direct sum. As already noted, the kernel of D are the harmonic forms. The theorem now follows form Lemma 119. For the last part, note that $\mathcal{H}^k \oplus \operatorname{im} d \subset \ker d$ is trivial. To see $\ker(d) \perp \operatorname{im}(d^*)$, note that $d\varphi = 0$ implies $\langle \varphi, d^*\omega\rangle = \langle d\varphi, \omega\rangle = 0$. ☐

It follows that $\mathcal{H}^k(M)$ is a complementary subspace of $\operatorname{im} d^{k-1}$ in $\ker d^k$. Hence the canonical map

$$\mathcal{H}^k(M,g) \hookrightarrow \ker(d^k) \twoheadrightarrow H^k_{dR}(M)$$

is an isomorphism. In particular, every cohomology class has a unique harmonic representative.

Corollary 122. $\dim H^k_{dR}(M) < \infty$.

Theorem 123 (Bochner)**.** *Let (M,g) be a closed oriented Riemannian manifold. Suppose $\mathrm{Ric}_g \geq 0$ and that there exists a point $p \in M$ with $\mathrm{Ric}_g(p) \neq 0$. Then $H^1_{dR}(M) = 0$.*

This is proven on exercise sheet 7 / 2. By verifying that the Hodge star operator preserves the harmonic forms, we see:

Corollary 124 (Poincaré duality)**.** *Let (M,g) be a closed oriented Riemannian manifold. Then the Hodge star operator restricts to an isomorphism*

$$*\colon \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M).$$

Hence $\dim H^k_{dR}(M) = \dim H^{n-k}_{dR}(M)$.

5 Asymptotics of the Heat Kernel

5.1 The Heat Equation

Let $S \to M$ be a Dirac bundle over a closed Riemannian manifold (M,g). The *heat equation* is the partial differential equation

$$\frac{\partial s}{\partial t} + D^2 s = 0.$$

Here we regard s as a family of sections $s_t \in C^\infty(S)$ for $t \geq 0$. It is required that s_t for $t > 0$ depends smoothly on t, while the dependence at $t = 0$ need only be continuous.

Proposition 125 (Existence and Uniqueness)**.** *Let $s_0 \in C^\infty(S)$. Then there is exists a unique solution (s_t) of the heat equation with given initial condition s_0.*

Proof. Uniqueness. For $t > 0$ we have

$$\frac{\partial}{\partial t}\|s_t\|^2 = \frac{\partial}{\partial t}\langle s_t, s_t\rangle = -\langle D^2 s_t, s_t\rangle - \langle s_t, D^2 s_t\rangle = -2\|Ds_t\|^2 \leq 0.$$

It follows that $\|s_t\| \leq \|s_0\|$ for all $t \geq 0$.

Existence. Let $s_t = e^{-tD^2}(s_0) = f_t(D)(s_0)$ where $f_t(x) = \exp(-tx^2)$. By formally taking the derivative, this is a solution of the heat equation. More precisely,

$$\left\| \frac{s_{t+h} - s_t}{h} + D^2 s_t \right\|_{W^k} \to 0 \quad (\text{for } h \to 0)$$

since $\|\frac{f_{t+h} - f_t}{h} + x^2 f_t\|_\infty \to 0$ and $(x^2 f_t)(D) = D^2 f_t(D)$. Recall here that $\|f(D)\|_{op} = \max_{\lambda \in \sigma(D)} |f(\lambda)|$ from the general theory of functional calculus. We now consider the behavior at $t = 0$. We have $\|f_t(D)s_0 - s_0\|_{W^k} \xrightarrow{t \to 0} 0$ for all k, so that $f_t(D)s_0 \to s_0$ in C^0. $\square$

Remark 126. *The proof of the theorem works already for $s_0 \in L^2(S)$ and (s_t) with C^1 in t and C^2 in $p \in M$. Such a solution of the heat equation (s_t) is then automatically smooth for $t > 0$.*

For $t > 0$ the operator $\exp(-tD^2)$ is a smoothing operator (exercise sheet 9/2). Hence we may write

$$(e^{-tD^2}s)(p) = \int_M k_t(p,q)s(q)d\mathrm{vol}(q). \tag{23}$$

for a smooth family $k_t \in C^\infty(S \boxtimes S^*)$ of smoothing kernels.

Proposition 127. *1. (k_t) is smooth in t and in (p,q). We have*

$$\frac{\partial}{\partial t}k_t + D_p^2 k_t = 0.$$

(where we apply the Dirac operator D_p only in p-direction.)

2. We have a C^0-convergent sequence of functions of p in $C^\infty(S)$

$$\int_M k_t(p,q)s(q)d\mathrm{vol}(q) \xrightarrow{t\to 0} s(p)$$

Thus $k_t(p,-) \xrightarrow{t\to 0} \delta_p$.

Proof. This is immediate by definition and Proposition 125. To see 1., we simply differentiate under the integral sign and since $s_t = \int_M k_t(-,q)s(q)$ is a solution with initial condition s_0, it converges in C^0 towards s_0 for $t \to 0$. □

The proposition in fact characterizes the smoothing kernel. Indeed, suppose that (K_t) is a family of smoothing kernels with properties 1. and 2. of Proposition 127. By 1. for all $t > \varepsilon > 0$ we have

$$K_t s = e^{-(t-\varepsilon)D^2} K_\varepsilon s$$

using also the uniqueness in Proposition 125. By property 2. we have $K_\varepsilon s \to s$ in C^0 for $\varepsilon \to 0$ and $e^{-(t-\varepsilon)D^2} \to e^{-tD^2}$ in the operator norm. It follows that $K_t s = e^{-tD^2}s$ for all $t > 0$.

5.2 Eigenvalue Growth of D

Recall from (22) that every $s \in L^2(S)$ may be decomposed $s = \sum_{\lambda\in\sigma(D)} s_\lambda$ into eigenvectors s_λ of D. We now prove the eigenvalue growth estimate used in the proof of Proposition 115. This estimate guarantees that the eigenvalues of D grow sufficiently fast, so that the series considered in the proof of Proposition 115 indeed converges.

Proposition 128. *Let $N(\Lambda)$ be the number of eigenvalues λ with modulus $|\lambda| \le \Lambda$, counted with multiplicity. We find a constant $C > 0$, depending only on M and the rank of S, with*

$$N(\Lambda) \le C \cdot (1+\Lambda)^{\frac{n(n+4)}{2}}.$$

Proof. Let $\varepsilon > 0$ and let $\{p_1, \dots, p_N\}$ be a maximal $\varepsilon/2$-net in M. This means that N is maximal with the property $B_{\varepsilon/2}(p_i) \cap B_{\varepsilon/2}(p_j) = \emptyset$ for $i \neq j$. Then $\bigcup_{i=1}^N B_\varepsilon(p_i) = M$ (for otherwise, we could introduce a new point x with $B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(p_i) = \emptyset$ for i). This argument also proves the existence of such nets (add new points until the equality $\bigcup_{i=1}^N B_\varepsilon(p_i) = M$ holds). Since M is compact, we have:

$$\exists r_0 > 0, c_0 > 0 \ \ \forall 0 \le r \le r_0 : \mathrm{vol}(B_r(p)) \ge c_0 r^n$$

Here $n = \dim M$. Hence

$$\mathrm{vol}(M) \ge \sum_{i=1}^N \mathrm{vol}(B_{\varepsilon/2}(p_i)) \ge Nc_0\left(\frac{\varepsilon}{2}\right)^n$$

and $N = N_\epsilon \leq c_1\varepsilon^{-n}$ where the constant c_1 depends only on M. This is an estimate for the number of points in a maximal $\varepsilon/2$-net.

Let $\lambda_1, \cdots, \lambda_k$ be the first k eigenvalues, in ascending order $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_k|$ (repeated according to their multiplicity). Let φ_i be the corresponding eigenvalues (chosen as an orthonormal basis of the eigenspace, if we have a multiple eigenvalue). Let $V = \sum_{i=1}^k \mathrm{Eig}(D, \lambda_i)$ (the eigenspace for a multiple eigenvalue contributes to this sum only once). We claim that for all $\varepsilon \leq \varepsilon_0$ we have an injective map

$$\chi\colon V \to S_{p_1} \oplus \cdots \oplus S_{p_N}, \quad \varphi \mapsto (\varphi(p_1), \ldots, \varphi(p_N)).$$

This will then lead to an estimate $\dim(V) \leq N\mathrm{rk}(S)$. To prove this, let $\varphi = \sum \alpha_i\varphi_i$ be an element of the kernel of χ. Suppose $x \in M$ and choose $p_i \in M$ with $d(x, p_i) < \varepsilon$. Then, using the Cauchy-Schwarz inequality and the compatibility of ∇ with the metric, we get

$$\|\varphi(x)\| = \|\varphi(x)\| - \|\varphi(p_i)\| = \int_0^1 \frac{d}{dt}\|\varphi(\gamma(t))\| dt \leq \varepsilon \|\nabla\varphi\|_{C^0}$$

for some curve $\gamma(t)$ from p_i to x of length $\leq \varepsilon$. Integrating over M gives

$$\|\varphi\|_{L^2}^2 \leq \varepsilon^2 \|\nabla\varphi\|_{C^0}^2 \mathrm{vol}(M). \tag{24}$$

For $l > n/2 + 1$ (e.g. $l = (n+4)/2$), the Sobolev Embedding Theorem gives an estimate $\|\nabla\varphi\|_{C^0} \leq \|\phi\|_{C^1} \leq c_2\|\varphi\|_{W^l}$ which we estimate further using the elliptic estimate

$$\|\varphi\|_{W^l} \leq c_3 \left(\|\varphi\|_{L^2} + \cdots + \|D^l\varphi\|_{L^2}\right) \leq c_4(1 + |\lambda|)^l \|\varphi\|_{L^2}$$

since $\|D^i\varphi\|_{L^2} \leq |\lambda_k|^i \|\varphi\|_{L^2}$ (λ_k is the largest eigenvalue). Putting this into (24) we get

$$\|\varphi\|_{L^2} \leq c_4\varepsilon(1 + |\lambda_k|)^{\frac{n+4}{2}} \sqrt{\mathrm{vol}(M)} \|\varphi\|_{L^2}.$$

If $\varepsilon \leq \frac{1}{2c_4\sqrt{\mathrm{vol}(M)}}(1 + |\lambda_k|)^{-\frac{n+4}{2}}$ we see $\|\varphi\|_{L^2} = 0$. Let $\varepsilon_0 = c_5(1 + |\lambda_k|)^{-\frac{n+4}{2}}$. With this ε_0 we have therefore shown that the map χ is injective. This implies $k = \dim(V) \leq N\mathrm{rk}(S)$ which combined with $N_{\varepsilon_0} \leq c_1\varepsilon_0^{-n} = c_6 \cdot (1 + |\lambda_k|)^{\frac{n(n+4)}{2}}$ proves the result. $\square$

5.3 Asymptotics of the Heat Kernel

Recall that in $\mathbb{R}^n$ with the standard metric the heat kernel (also called the *fundamental solution* to the heat equation) is given by

$$k_t(p, q) = \frac{1}{(4\pi t)^{n/2}} \exp(-d(p, q)^2/(4t)),$$

as may be verified directly by putting it into the heat equation $\frac{\partial}{\partial t}\varphi + \Delta\varphi = 0$ (where the Laplacian is acting on the p-variable of k_t) and computing the limit $\lim_{t\mapsto 0} \int_{\mathbb{R}^n} k_t(p, q) f(q) dvol_{\mathbb{R}^n}(q) = f(p)$ for a compactly supported $f \in C^\infty(\mathbb{R}^n)$. On a general Riemannian manifold (M, g) we define an approximation of the heat kernel by the same formula ("Euclidean heat kernel")

$$h_t(p, q) = \frac{1}{(4\pi t)^{n/2}} \exp(-d(p, q)^2/(4t)),$$

using now the geodesic distance d on M. The maps h_t are defined for $t > 0$ and are smooth in a neighborhood of points (p, q) of the diagonal in $M \times M$. We wish to approximate the actual heat kernel k_t in terms of h_t.

We begin by computing $(\frac{\partial}{\partial t} + \Delta)h_t(-, q)$ where now $\Delta = d^*d$ is the connection Laplacian (also called *Laplace-Beltrami operator* on $C^\infty(M)$, the smooth real valued functions on M. Fix $q \in M$ and consider $h_t = h_t^{(q)} = h_t(-, q)$ in local geodesic normal coordinates $(x^1, \ldots, x^n)$ around q. Let $r^2 = (x^1)^2 + \cdots + (x^n)^2$. Then

$$h_t(-, q) = \frac{1}{(4\pi t)^{n/2}} \exp(-r^2/(4t)).$$

Lemma 129. *We have* $\left(\frac{\partial}{\partial t}+\Delta\right)h_t(-,q)=\frac{r\cdot h_t(-,q)}{4gt}\frac{\partial g}{\partial r}$ *for* $g=\det\left(g_{ij}(-)\right)$.

Proof. Let $\Delta=\nabla^*\nabla$ be the connection Laplacian on functions. First, for the gradient (identifying $T^*M=TM$) we find

$$\nabla h_t=-\frac{h_t}{2t}r\partial_r$$

so

$$\Delta h_t=-\frac{h_t}{2t}\nabla^*(r\partial_r)+\frac{r}{2t}\frac{\partial h_t}{\partial r} \tag{25}$$

where we use that $\nabla^*(fX)=f\nabla^*(X)-df(X)$ for $f\in C^\infty(M), X\in C^\infty(TM)$. Using (15) we find

$$\nabla^*(r\partial_r)=-\mathrm{div}(r\partial_r)=-\frac{1}{\sqrt{g}}\partial_j(x^j\sqrt{g})=-n-\frac{r}{2g}\frac{\partial g}{\partial r}$$

where we use $\partial_r=r^{-1}x^j\partial_j$. Putting this into (25) gives

$$\begin{aligned}\Delta h_t&=\frac{-h_t}{2t}\left(-n-\frac{r}{2g}\frac{\partial g}{\partial r}\right)+\frac{r}{2t}\frac{\partial h_t}{\partial r}\\&=\frac{-h_t}{2t}\left(-n-\frac{r}{2g}\frac{\partial g}{\partial r}\right)-\frac{r^2h_t}{4t^2}\end{aligned}$$

On the other hand,

$$\frac{\partial h_t}{\partial t}=\left(\frac{-n}{2t}+\frac{r^2}{4t^2}\right)h_t.$$

□

Keeping q fixed, we now make the following formal ansatz for the actual heat kernel k_t:

$$k_t(p,q)=h_t(p,q)\left(\Theta_0(p,q)+t\Theta_1(p,q)+t^2\Theta_2(p,q)+\cdots\right), \tag{26}$$

where $\Theta_j\in C^\infty(S\boxtimes S^*)$ for $j=0,1,\ldots$. We have the following product rule for the square of the Dirac operator from [Roe, Lemma 7.13], which can be derived using synchronous orthonormal frames.

Lemma 130. *For* $h\in C^\infty(M)$ *and* $s\in C^\infty(S)$ *we have*

$$D^2(h\cdot s)=hD^2s+(\Delta h)\cdot s-2\nabla_{\nabla h}s.$$

Combining the lemma with the above calculations for h_t, we then get for $s\in C^\infty(S\boxtimes S_q^*)$ (note that for fixed $v\in S_q$ this is just a smooth section of S)

$$\left(\frac{\partial}{\partial t}+D^2\right)(h_t\cdot s)=h_t\left(\frac{\partial}{\partial t}+D^2\right)s+\frac{rh_t}{4gt}\frac{\partial g}{\partial r}s+\frac{h_t}{t}\nabla_{r\partial r}s. \tag{*}$$

Now formally write $s=u_0+tu_1+t^2u_2+\ldots$ where $u_j(p)=u_j^{(q)}(p):=\Theta_j(p,q)$. For $s=t^ku_j$ the right-hand side of (*) is

$$h_t\left(jt^{j-1}u_j+t^jD^2u_j+\frac{r}{4g}\frac{\partial g}{\partial r}t^{j-1}u_j+t^{j-1}\nabla_{r\partial_r}u_j\right)$$

For $s=t^{j-1}u_{j-1}$ the right-hand side of (*) is

$$h_t\left((j-1)t^{j-2}u_{j-1}+t^{j-1}D^2u_{j-1}+\frac{r}{4g}\frac{\partial g}{\partial r}t^{j-2}u_{j-1}+t^{j-2}\nabla_{r\partial_r}u_{j-1}\right)$$

The coefficients of the terms with exponent t^{j-1} sum up to

$$D^2u_{j-1}+\nabla_{r\partial r}u_j+\left(j+\frac{r}{4g}\frac{\partial g}{\partial r}\right)u_j$$

If k_t is a solution to the heat equation, these should all vanish. Beginning with $u_{-1} = 0$ we get a recursive system of coefficients. A system of solutions (u_j) then gives a formal candidate $k_t = h_t(u_0 + tu_1 + t^2u_2 + \ldots)$ for the heat kernel. We therefore turn to the question of solving recursively the ordinary differential equations

$$D^2u_{j-1} + \nabla_{r\partial r}u_j + \left(j + \frac{r}{4g}\frac{\partial g}{\partial r}\right)u_j = 0 \tag{**}$$

Introduce the 'integrating factor' $r^j g^{1/4}$. Given u_{j-1}, a solution u_j of (**) needs to satisfy

$$\begin{aligned}
\nabla_{\partial_r}\left(r^j g^{1/4}u_j\right) &= jr^{j-1}g^{1/4}u_j + r^j\frac{1}{4}g^{-3/4}\frac{\partial g}{\partial r}u_j + r^j g^{1/4}\nabla_{\partial_r}u_j \\
&= r^{j-1}g^{1/4}\left(ju_j + \frac{r}{4g}\frac{\partial g}{\partial r}u_j + \nabla_{r\partial_r}u_j\right) \\
&= \begin{cases} 0 & (j = 0) \\ -r^{j-1}g^{1/4}D^2u_{j-1} & (j > 0). \end{cases}
\end{aligned}$$

We solve this equation for $u_j(r) \in C^\infty(S_p \boxtimes S_q^*) \cong \mathrm{End}(S_q)$ (the identification uses parallel transport along a radial geodesic from q to p) on geodesic rays, beginning at q. The solution for the first such equation is

$$r^0 g^{1/4}u_0 = \text{const}$$

for some constant, for which we choose id_{S^q}. For the higher solutions u_j we must take

$$u_j = -r^{-j}g^{-1/4}\int_0^r \rho^{j-1}g^{1/4}(\rho)D^2u_{j-1}d\rho \in C^\infty(S \boxtimes S_q^*). \tag{27}$$

(the integration constant zero is determined by the requirement that u_j extends to a smooth function at $r = 0$.) Note that each u_j is only defined in a neighborhood of the diagonal in $M \times M$.

We have thus shown:

Proposition 131. *For $\Theta_j(p,q) = u_j^{(q)}(p)$ we have a (formal) solution*

$$h_t(-,q)\left(\sum_{j=0}^{\infty} t^j\Theta_j(-,q)\right)$$

of the heat equation for p close to q and $t > 0$. With the requirement $\Theta_0(q,q) = \mathrm{id}_{S_q}$ for all $q \in M$ the sequence (Θ_j) of smooth sections in $C^\infty(S \boxtimes S^*)$ (which is defined in a neighborhood of the diagonal in $M \times M$) is uniquely determined. The functions Θ_j depend only on g and the covariant derivative ∇^S and can (in principle) be determined recursively by solving the above ordinary differential equation.

What is the relationship with the actual solution $k_t(-,q)$ of the heat equation? We claim that it is an *asymptotic development* of the heat kernel: the ℓ-th partial sum

$$k_t^\ell(p,q) = h_t(p,q)\left(\sum_{j=0}^{\ell} t^j\Theta_j(p,q)\right)$$

has the following property: For all $\nu, m > 0$ there exists $\ell(\nu, m)$ so that for all $\ell \geq \ell(\nu, m)$ we have the following estimate for the maximum norm of (at most) m-th derivates

$$\left\|k_t(p,q) - k_t^\ell(p,q)\right\|_{C^m} \leq C_{\ell,\nu,m}|t|^\nu$$

for small $|t|$ and some constant $C_{\ell,n,m}$. We may thus view the sequence $(k_t^\ell)_{\ell\geq 0}$ as a sort of *Taylor expansion* of the heat kernel. But note that we are not claiming (as little as for usual Taylor expansions of smooth functions $\mathbb{R}\to\mathbb{R}$) that for $t>0$ this sequence is convergent in $C^\infty(S\boxtimes S^*)$.

We will henceforth write

$$h_t\cdot\sum_{j=0}^{\infty}t^j\Theta_j \overset{t\to 0}{\sim} k_t$$

for this asymptotic behavior.

In the following we extend all Θ_j arbitrarily to smooth sections in $C^\infty(S\boxtimes S^*)$ defined on the whole of $M\times M$. In particular the partial sums $k_t^\ell\in C^\infty(S\boxtimes S^*)$ are defined on $M\times M$. We will prove the above estimates $\left\|k_t-k_t^\ell\right\|_{C^m}\leq C_{\ell,\nu,m}|t|^\nu$, $|t|\leq|t_0|$, for these extended k_t^ℓ. We will see later that the essential information of the asymptotic development of the heat kernel is in fact determined by the Θ_j around the diagonal of $M\times M$, but for the following computation it is convenient to have k_t^ℓ defined on the whole of $M\times M$.

For the proof of the asymptotic behavior of k_t^ℓ we examine to what extent k_t^ℓ enjoys the characteristic properties of a heat kernel. This is done in two steps.

1.) $k_t^\ell(p,-)\overset{t\to 0}{\to}\delta_p$, uniformly in p.

Proof: For fixed p we split the relevant integral into two parts, over a ball $B_r(p)$ of (small) radius r, and its complement

$$\int_M k_t^\ell(p,q)s(q)\mathrm{dvol}(q)=\int_{B_r(p)}k_t^\ell s(q)\mathrm{dvol}(q)+\underbrace{\int_{M\setminus B_r(p)}k_t^\ell s(q)\mathrm{dvol}(q)}_{\xrightarrow{t\to 0}0}$$

where we use the fact that for $p\neq q$ the heat kernel $h_t(p,q)$ is rapidly decreasing for $t\to 0$. For the first summand we get

$$\int_{B_r(p)}k_t^\ell s(q)\mathrm{dvol}(q)=\underbrace{\int_{B_r(q)}h_t(p,q)\Theta_0(p,q)s(q)\mathrm{dvol}(q)}_{\xrightarrow{t\to 0}s(p)+O(r)}+\sum_{j=1}^{\ell}\underbrace{\int_{B_r(q)}h_t(p,q)\Theta_j(p,q)t^js(q)\mathrm{dvol}(q)}_{\xrightarrow{t\to 0}0}$$

using the above calculation $\Theta_0(p,q)=\mathrm{id}+O(\|p-q\|)\in\mathrm{Hom}(S_q,S_p)$ and the fact that the Euclidean heat kernel $h_t(p,-)$ converges to δ_p for $t\to 0$. Letting r go to zero shows the claim.

2.) $\left(\frac{\partial}{\partial t}+D^2\right)k_t^\ell=\begin{cases}t^\ell h_te_t^\ell & \text{near the diagonal, for some } e_t^\ell\in C^\infty(S\boxtimes S^*),\\ h_t\frac{1}{t}\gamma_t & \text{outside a neighborhood of the diagonal, for some } \gamma_t\in C^\infty(S\boxtimes S^*)\text{, by (*).}\end{cases}$

For $\ell>\nu+k+n/2$ we have $t^{-\nu}(t^\ell h_te_t^\ell)\to 0$ in C^k for $t\to 0$. Moreover, $t^{-\nu}\left(\frac{1}{t}h_t\gamma_t\right)\to 0$ in C^k for $t\to 0$ since h_t is rapidly decreasing outside a neighborhood of the diagonal. In any case, we have shown $\forall\nu,k\ \exists\ell(\nu,k)$ so that for all $\ell>\ell(\nu,k)$ we may write

$$\left(\frac{\partial}{\partial t}+D^2\right)k_t^\ell=t^\nu r_t^\ell(p,q)\in O(|t|^\nu),$$

for $r_t\in C^\infty(S\boxtimes S^*)$, $r_0=0$, and r_t is continuous in $t\geq 0$ for the C^k-topology.

Now consider $s_t^\ell(-,q)=k_t-k_t^\ell$. The left hand side is a solution of the inhomogenous heat equation

$$\left(\frac{\partial}{\partial t}+D^2\right)s_t=-t^\nu r_t(-,q)=:\tilde{s}_t,\qquad s_0=0.$$

On the other hand, such a solution s_t may always be written $s_t=\int_0^t e^{-(t-\tau)D^2}\tilde{s}_\tau d\tau$. Since $k_t^\ell+s_t\to\delta_p$ for $t\to 0$ as shown in part 1.) and by the uniqueness of solutions of the (homogenous) heat equation (which is solved by $k_t^\ell+s_t$), we get

$$k_t^\ell(p,q)+s_t(p,q)=k_t(p,q).$$

Also, since $\tilde{r}_t \to 0$ for $t \to 0$ in the C^k-maximum norm, and hence in the W^k-Sobolev norm, we have

$$\|s_t\|_{W^k} \leq t \cdot C_k \cdot \sup\{\|\tilde{s}_\tau\|_{W^k} \mid 0 \leq \tau \leq t\} \leq C'_k |t|^{\nu+2}$$

because $e^{-(t-\tau)D^2}$ is bounded in all Sobolev norms. By the Sobolev Embedding Theorem we finally conclude $\|s_t\|_{C^m} \leq |t|^{\nu+2}$ for $|t| \leq t_0$ for $k > m + n/2$ finishing the proof of the asymptotic behavior of k_t^ℓ.

We now determine the first Θ_j. Recall that we work in normal coordinates centered to q, which is held fixed. We have already seen that $\Theta_0(p,q)\colon S_q \to S_p$ is given by parallel transport along the geodesic ray from q to p and then multiplying with the factor $g^{-1/4}(p)$ (recall that $g(p) = \det(g_{ij}(p))$. To put this into a formula, we work in the following frame. Let $(f_1, \ldots, f_d)$ by an orthonormal basis of S_q. We may extend these along geodesic rays emanating at q to get an orthonormal frame $f_i(p)$ on our chart neighborhood. Then

$$u_0^{(q)}(p) = \Theta_0(p,q)\left(v^j f_j(q)\right) = g^{-1/4}(p) v^j f_j(p), \quad v^j \in \mathbb{R}. \tag{28}$$

Recall from our earlier discussion (27) that

$$u_1(r) = -\frac{1}{r} g^{-1/4}(r) \int_0^r g^{1/4}(\rho) D^2 u_0 \, d\rho.$$

Hence $\Theta_1(q,q) = u_1(0) = -(D^2 u_0)(0)$. It remains to compute $D^2 u_0$. For fixed q and $v = v^j f_j(q) \in S_q$ we may view $u_0(v)$ as a section $s \in C^\infty(S)$ by $s(p) = \Theta_0(p,q)v$. Applying Lemma 130 to (28) we get

$$D^2 s(q) = v^j \left(g^{-1/4} D^2(f_j) + \Delta(g^{-1/4}) \cdot f_j - 2\nabla_{\nabla g^{-1/4}} f_j\right)\Big|_q \tag{29}$$

The covariant derivatives of f_j all vanish in q, so the last term is zero. By the Weitzenböck formula (Theorem 33) we have

$$D^2(f_j)(q) = K(q)(f_j),$$

because $\Delta(f_j)(q) \overset{(11)}{=} -\sum_i \nabla_{\partial_i, \partial_i}(f_j) = 0$ (note that $\nabla_{\partial_i} f_j$ is constantly zero along the coordinate axis x^i by construction of f_j).

Moreover, in normal coordinates from (7) we get

$$g = \det(g_{ij}) = 1 - \frac{1}{3}\mathrm{Ric}_{kl}(q) x^k x^l + O(r^3)$$

so

$$g^{-1/4} = 1 + \frac{1}{12}\mathrm{Ric}_{kl}(q) x^k x^l + O(r^3).$$

This gives

$$\Delta(g^{-1/4})(q) = -\sum \frac{\partial^2}{\partial x_i^2} g^{-1/4}(q) = -\frac{1}{6}\mathrm{scal}_g(q).$$

Here we recall that by equation (15) the laplacian of a function $f : M \to \mathbb{R}$ in local coordinates around q is given by

$$\Delta(f) = -\mathrm{div}(\mathrm{grad} f) = -\frac{1}{\sqrt{g}} \partial_k (\sqrt{g} g^{kj} \frac{\partial f}{\partial x^j})$$

so that the calculation is justified by the expansion of $g^{-1/4}$ around q stated above and the fact that $g^{kj}(q) = \delta^{kj}$.

Now we have evaluated all terms in (29). In summary,

$$u_1(0) = -D^2 s(q) = \frac{1}{6}\mathrm{scal}_g(q)\,\mathrm{id}_{S_q} - K(q)$$

Example 132. *In the special case $S = \Lambda^* T^* M$ we know that $D^2 = d^* d + d d^*$ is the Hodge Laplacian, which restricted to $C^\infty(M)$ is the usual Laplace-Beltrami operator Δ and*

$$\Theta_0(q,q) = \mathrm{id}_{S_q}, \quad \Theta_1(q,q) = \frac{1}{6}\mathrm{scal}_g(q)\,\mathrm{id}_{S_q}.$$

Next, on $C^\infty(\Lambda^1 T^* M)$ we have $\Theta_0(q,q) = \mathrm{id}$, $\Theta_1(q,q) = \frac{1}{6}\mathrm{scal}_g(q) - \mathrm{Rc}(q)$ for the Ricci endomorphism.

5.4 Spectral Geometry

Let

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots$$

be the eigenvalues of D^2, counted with multiplicity. We define the *trace* of e^{-tD^2} as

$$\operatorname{tr}(e^{-tD^2}) := \sum_{i=0}^{\infty} e^{-t\lambda_i}.$$

Using Exercise sheet 11 we have the equality

$$\sum_{i=0}^{\infty} e^{-t\lambda_i} = \int_M \operatorname{tr}(k_t(p,p))d\mathrm{vol}(p). \tag{30}$$

with the heat kernel (k_t) of e^{-tD^2}. Using the asymptotic expansion of k_t and noting that for the Euclidean heat kernel we have $h_t(p,p) = \frac{1}{(4\pi t)^{n/2}}$ we obtain

$$\operatorname{tr}(e^{-tD^2}) \overset{t\to 0}{\sim} \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} t^j \int_M \operatorname{tr}(\Theta_j(p,p))d\mathrm{vol}(p). \tag{31}$$

In the special case of the Laplace-Beltrami operator Δ on $C^\infty(M)$ the previous computation implies

$$\sum_{i=0}^{\infty} e^{-t\lambda_i} \overset{t\to 0}{\sim} \frac{1}{(4\pi t)^{n/2}} \left(\mathrm{vol}(M) + \frac{1}{6}\int_M \mathrm{scal}_g(p)d\mathrm{vol}(p)\cdot t + O(t^2)\right).$$

It follows that the spectrum of Δ determines the dimension n, the volume and the *total scalar curvature* of M.

The spectrum of Δ does *not* determine M up to isometry! Indeed there exist non-isometric manifolds (M,g) and (M',g') which are *isospectral*, i.e. whose Laplacians have identical eigenvalues, counted with multiplicity. The first such examples are due to Milnor, see also Mark Kac's article: "Can one hear the shape of a drum?" and the corresponding Wikipedia article. The question which properties of (M,g) are determined by the spectrum of the Laplace operator is treated by the subject called *spectral geometry.* In physical language one asks which properties of a (geometric) object are determined by the spectrum of its emitted radiation.

Remark 133. *For the Laplace-Beltrami operator Δ on $C^\infty(M)$ the number of eigenvalues $N(\Lambda)$ less than Λ is given asymptotically by $N(\Lambda) \sim \frac{1}{(4\pi)^{n/2}\Gamma(n/2+1)}\mathrm{vol}(M)\Lambda^{n/2}$ (this means that the quotient converges to 1). This famous result, which sharpens our Proposition 128 is called* Weyl asymptotics.

6 The Index Theorem

6.1 The Index of Graded Dirac Operators

Let (M^n,g) be a compact Riemannian manifold equipped with a Dirac bundle $(S,\nabla^S)\to M$ (real or complex). Recall from Definition 31 that the corresponding Dirac operator $D\colon C^\infty(S)\to C^\infty(S)$ is given by

$$D(s) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}(s)$$

for a (local) orthonormal frame e_i of TM.

Definition 134. *A $\mathbb{Z}/2$-grading on S is a bundle endomorphism $\varepsilon\colon S\to S$ satisfying*

1. $\varepsilon^2 = \mathrm{id}$, *this means* ϵ *is an involution.*

2. ε *is self-adjoint, that is,* $(\varepsilon s_1, s_2) = (s_1, \varepsilon s_2)$ *for* $s_1, s_2 \in S_x$, $x \in M$.

3. ε *anti-commutes with Clifford-multiplication, so* $\varepsilon(v \cdot s) = -v \cdot \varepsilon(s)$ *for* $v \in C^\infty(TM)$ *and* $s \in C^\infty(S)$.

4. ε *is parallel, i.e., we have* $\nabla^S_v(\varepsilon\, s) = \varepsilon\left(\nabla^S_v s\right)$

Given such a grading we get an orthogonal decomposition $S = S_+ \oplus S_-$ into (± 1)-eigenspaces of ε (here we use 1. and 2.). The Dirac operator restricts maps $D_\pm \colon C^\infty(S_\pm) \to C^\infty(S_\mp)$, by 3. and 4. Conversely, such an orthogonal decomposition $S = S_+ \oplus S_-$ defines a $\mathbb{Z}/2$-grading, provided the connection preserves each of the bundles $S_\pm$ and Clifford multiplication exchanges S_+ and S_-. The map ε is then given by $\varepsilon|_{S_\pm} = \pm\,\mathrm{id}$. For this decomposition the Dirac operator splits as

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}. \tag{32}$$

In particular, $D_+^* = D_-$ are adjoints of each other. Moreover, $\ker(D) = \ker(D_+) \oplus \ker(D_-)$ which is an orthogonal decomposition into finite-dimensional subspaces $\ker(D_\pm) \subset C^\infty(S_\pm)$.

Example 135. 1. *Let* $S = \Lambda^*(T^*M)$. *Recall from* (13) *that the corresponding Dirac operator is the deRham operator* $D = d + d^* \colon \Omega^*(M) \to \Omega^*(M)$, *using the notation* $\Omega^*(M) = \bigoplus_k \Omega^k(M)$ *and* $\Omega^k(M) = C^\infty(M, \Lambda^k(T^*M))$. *On the Dirac bundle* S *we define* $\varepsilon(w) = (-1)^{\deg(w)} w$ *so that*

$$S_+ = \bigoplus_{k\ even} \Lambda^k(T^*M), \quad S_- = \bigoplus_{k\ odd} \Lambda^k(T^*M).$$

The corresponding Dirac operator $D_+ \colon \Omega^{ev}(M) \to \Omega^{odd}(M)$ *is called the* Euler operator.

2. *Let* $S = \Lambda^*(T^*M) \otimes \mathbb{C}$. *We then again have a grading* $S = \Lambda^{ev} \oplus \Lambda^{odd}$, *as in the last example. However, there exists a second grading when* $\dim M = n$ *is even and* M *is oriented.*

   For an orthonormal frame $(e_1, \dots, e_n)$ *of* TM *the* volume element *is given by* $\omega = e_1 \cdots e_n \in C^\infty(\mathrm{Cl}(TM))$ (*exercise sheet 3/2*). *Note that this defined a global form on* M *because* M *is oriented. Recall also* $\omega^2 = (-1)^{\frac{n(n+1)}{2}}$ *and* $v\omega = (-1)^{n-1}\omega v$ *for* $v \in C^\infty(TM)$.

   We now define a grading by $\varepsilon(s) = i^m \omega \cdot s$, *where* $m = n/2$. *This is a grading since* $\varepsilon^2 = 1$ *and since Clifford multiplication with* ω *is self-adjoint* (*as* n *is even*). *We have*

$$\omega s = (-1)^{\frac{k(k-1)}{2}} * (s), \qquad s \in \Lambda^k T^*M \otimes \mathbb{C}.$$

   This follows since $v \cdot s = v \wedge s - \iota_v s$ (*Proposition 22*):

$$\begin{aligned} e_1 \cdots e_n \cdot (e_1 \wedge \cdots \wedge e_k) &= (-1)^{\frac{n(n-1)}{2}} e_n \cdots e_1 (e_1 \wedge \cdots \wedge e_k) \\ &= (-1)^{\frac{n(n-1)}{2}} (-1)^k e_n \cdots e_{k+1} \cdot 1 = (-1)^{\frac{n(n-1)}{2}} (-1)^k (-1)^{\frac{(n-k)(n-k-1)}{2}} e_{k+1} \cdots e_n \cdot 1 \\ &= (-1)^{\frac{n(n-1)+2k+(n-k)(n-k-1)}{2}} e_{k+1} \wedge \cdots \wedge e_n = (-1)^{\frac{k(k-1)}{2}} * (e_1 \wedge \cdots \wedge e_k) \end{aligned}$$

   The corresponding Dirac operator $D = d + d^* \colon C^\infty(S_+) \to C^\infty(S_-)$ *is called the* signature operator (*so the bundles* $S_\pm$ *are the* ± 1*-eigenbundles of the operator* $\varepsilon = i^{m+k(k-1)} *$).

3. *Let* $(M^{2m=n}, g)$ *be a spin manifold with corresponding spinor bundle* $S \to M$. *Recall that this bundle is associated to the complex representation* Δ *of* $\mathrm{Cl}(2m)$ (*exercise sheet 5/3*), *i.e.* $S = P_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}(n)} \Delta$. *We recall that when restricted to* $\mathrm{Spin}(2m)$ *this representation splits into the* (*only*) *two irreducible representations* $\Delta = \Delta_+ \oplus \Delta_-$. *Clifford multiplication* $v \in \mathbb{R}^n$ *interchanges* $\Delta_\pm$. *Hence*

$$S = S_+ \oplus S_-, \qquad S_\pm = P_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}(n)} \Delta_\pm$$

   and we obtain a corresponding Dirac operator $D_\pm \colon C^\infty(S_\pm) \to C^\infty(S_\mp)$. *This is the so-called* spinor Dirac operator, *also called* Atiyah-Singer operator, *compare the end of Section 3.*

Definition 136. *Let $S = S_+ \oplus S_- \to M$ be $\mathbb{Z}/2$-graded Dirac bundle with corresponding Dirac operator D. The* index *of D is defined as*

$$\operatorname{ind}(D) = \dim\ker D_+ - \dim\ker D_- \in \mathbb{Z}.$$

(*taking the real or complex dimension, according to the case at hand.*) *Recall here that by elliptic regularity these kernels are finite-dimensional.*

Remark 137. *Note that the index is equal to $\dim\ker(D_+) - \dim\ker(D_+^*)$. Without grading, the index $\dim\ker(D) - \dim\ker(D^*)$ would automatically vanish, since D is self-adjoint.*

Example 138.

1. *For the Euler operator we have $S_+ = \Omega^{ev}$, $S_- = \Omega^{odd}$ and $D = d + d^*$. It follows that $\ker D_+ = \bigoplus_{k\ even} \mathcal{H}^k(M)$ are the even harmonic forms on M. Hence $\ker D_+ = H^{ev}_{dR}(M)$ by Hodge theory. Similarly, $\ker D_- = H^{odd}_{dR}(M)$. Hence we obtain the* Euler characteristic *as index*

$$\operatorname{ind}(D) = \dim H^{ev}(M) - \dim H^{odd}(M) = \chi(M).$$

This is a purely topological invariant of M, in particular it is independent of the metric.

2. *For the signature operator on an oriented $4l = 2m = n$-dimensional manifold we have*

$$\varepsilon = i^{m+k(k-1)}*\colon \Lambda^k(T^*M)\otimes\mathbb{C} \to \Lambda^{n-k}(T^*M)\otimes\mathbb{C}$$

Consider $0 \le k \le n$ with $k \ne n/2$. We consider ε on $(\alpha,\beta) \in \Lambda^k \oplus \Lambda^{n-k}$. Because $\epsilon^2 = 1$, we have

$$\varepsilon(\alpha,\beta) = (\alpha,\beta) \Leftrightarrow \varepsilon\alpha = \beta.$$

Hence $\ker\left(D_+|_{\Lambda^k\oplus\Lambda^{n-k}}\right) = \{\omega \in \Omega^k(M)\otimes\mathbb{C} \mid (d+d^*)\omega = 0\} = \mathcal{H}^k(M;\mathbb{C})$ since $*$ preserves harmonic forms (given $\omega \in \mathcal{H}^k(M;\mathbb{C})$, the element $(\omega, *\omega) \in \ker\left(D_+|_{\Lambda^k\oplus\Lambda^{n-k}}\right)$ lies in the kernel). Similarly, $\ker\left(D_-|_{\Lambda^k\oplus\Lambda^{n-k}}\right) = \mathcal{H}^k(M;\mathbb{C})$. It follows that

$$\operatorname{ind}(D) = \dim\ker D_+|_{\Lambda^m} - \dim\ker D_-|_{\Lambda^m}$$

is concentrated in the middle dimension m. For $\omega \in \Omega^m$ we have $\varepsilon\omega = i^{m+m(m-1)} * \omega = *\omega$ since m is even. Hence $\ker D_\pm|_{\Lambda^m} = \{\omega \in \mathcal{H}^m(M) \mid *\omega = \pm\omega\}$ is given by (anti)-self-dual forms. These are related to the signature of M, defined as the signature of the non-degenerate symmetric bilinear form

$$\psi\colon (\alpha,\beta) \mapsto \int_M \alpha\wedge\beta$$

(*exercise sheet 10/4*). *In a suitable basis, ψ takes the form*

$$\psi = \begin{pmatrix} E_r & 0 \\ 0 & -E_s \end{pmatrix}$$

and the signature of M is $r - s$. Now for a self-dual $\omega \in \mathcal{H}^m_+(M)$ with $\|\omega\| = 1$ we have

$$\int_M \omega\wedge *\omega = \int_M (\omega,\omega)d\mathrm{vol} = 1,$$

using the definition of the Hodge operator (Definition 23). Similarly, $\psi(\omega,\omega) = -1$ for a harmonic anti-self-dual form. Hence the index is the signature

$$\operatorname{ind}(D) = \dim\mathcal{H}^m_+(M) - \dim\mathcal{H}^m_-(M) = \operatorname{sign}(M),$$

which is again a purely topological invariant.

3. *For the spinor Dirac operator, what is* $\mathrm{ind}(D)$? *Is it also a topological quantity, as in the previous two examples? The answer is yes. Indeed, the Atiyah-Singer Index Theorem (Atiyah-Singer, 1968) expresses* $\mathrm{ind}(D)$ *in terms of topological quantities (for any Dirac operator* D*). It was first proven using topological* K*-theory and later (Atiyah-Bott-Patodi, 1973) by using the asymptotics of the heat kernel, which is the approach we shall follow.*

Given a general Dirac operator $D_\pm\colon C^\infty(S_\pm) \to C^\infty(S_\mp)$, we consider $D_+^2 := D_- \circ D_+\colon C^\infty(S_+) \to C^\infty(S_+)$ and similarly $D_-^2 = D_+ \circ D_-\colon C^\infty(S_-) \to C^\infty(S_-)$. Hence

$$D^2 = D_+^2 \oplus D_-^2\colon C^\infty(S_+) \oplus C^\infty(S_-) \to C^\infty(S_+) \oplus C^\infty(S_-).$$

The eigenspace decomposition of D^2 induces corresponding eigenspace decompositions for D_+^2, D_-^2 (note that the grading operator ϵ commutes with D^2):

$$\mathrm{Eig}(D^2, \lambda) = \mathrm{Eig}(D_+^2, \lambda) \oplus \mathrm{Eig}(D_-^2, \lambda).$$

For $\lambda = 0$, we have $D_+^2 s = 0$ precisely when $D_+ s = 0$. The non-trivial implication follows from $0 = \langle D_+^2 s, s\rangle = \langle D_+ s, D_+ s\rangle$. Hence

$$\mathrm{Eig}(D_\pm^2, 0) = \ker D_\pm^2 = \ker D_\pm \tag{33}$$

We know that all eigenvalues λ of D^2 are non-negative. We have already considered the case $\lambda = 0$, so suppose $\lambda > 0$. The operator D_+ induces a map

$$D_+\colon \mathrm{Eig}(D_+^2, \lambda) \to \mathrm{Eig}(D_-^2, \lambda) \tag{34}$$

since $D_-^2 D_+(s) = D_+ D_- D_+(s) = D_+ D_+^2(s) = \lambda(D_+ s)$ for $s \in \mathrm{Eig}(D_+^2, \lambda)$. Similarly,

$$D_-\colon \mathrm{Eig}(D_-^2, \lambda) \to \mathrm{Eig}(D_+^2, \lambda). \tag{35}$$

The composition of (34) with (35) is just multiplication by $\lambda\colon \mathrm{Eig}(D_+^2, \lambda) \to \mathrm{Eig}(D_+^2, \lambda)$. It follows that

$$\mathrm{Eig}(D_+^2, \lambda) \cong \mathrm{Eig}(D_-^2, \lambda).$$

Definition 139. *We define*

$$\mathrm{tr}\, e^{-tD_\pm^2} := \sum_{\lambda_i^\pm} e^{-t\lambda_i^\pm}$$

as the trace *of the operators* $e^{-D_\pm^2}$. *(Note that the right hand side is absolutely convergent). The* super trace *of the* $\mathbb{Z}/2$*-graded operator* D^2 *is defined as*

$$\mathrm{tr}_S\, e^{-tD^2} = \mathrm{tr}\, e^{-tD_+^2} - \mathrm{tr}\, e^{-tD_-^2}.$$

Note that by (32) the trace of e^{-tD^2} is $\mathrm{tr}\, e^{-tD^2} = \mathrm{tr}\, e^{-tD_+^2} + \mathrm{tr}\, e^{-tD_-^2}$. In this respect the supertrace is a $\mathbb{Z}/2$-graded version of the absolute trace.

Corollary 140. *Let* $0 \le \lambda_0^+ \le \lambda_1^+ \le \cdots$ *and* $0 \le \lambda_0^- \le \lambda_1^- \le \cdots$ *be the eigenvalues of* D_+^2 *and* D_-^2*, respectively. All non-zero eigenvalues agree and occur with the same multiplicity* (*the multiplicity of the eigenvalue zero may differ*). *Together with* (33) *we obtain the McKean-Singer formula* (1967)

$$\mathrm{tr}_S\, e^{-tD^2} = \mathrm{tr}\, e^{-tD_+^2} - \mathrm{tr}\, e^{-tD_-^2} = \sum_{\lambda_i^+} e^{-t\lambda_i^+} - \sum_{\lambda_i^-} e^{-t\lambda_i^-} = \dim\ker D_+ - \dim\ker D_- = \mathrm{ind}(D).$$

Recall that e^{-tD^2} is the time evolution operator for the heat equation, so that $s_t = e^{-tD^2} s_0$ is a solution of the heat equation with initial condition s_0.

For $t \to \infty$ the operator $e^{-tD_\pm^2}$ converges towards the projection onto the kernels of $D_\pm^2$ (in the operator norm for L^2). Indeed, applying functional calculus to D^2 and the function $e^{-t\lambda^2}$ shows that only the non-zero eigenvalues survive in the (time) limit under the heat evolution. Hence

$$\operatorname{tr}_S e^{-tD^2} \xrightarrow{t\to\infty} \operatorname{tr} \pi_{\ker D_+^2} - \operatorname{tr} \pi_{\ker D_-^2}$$

for the orthogonal projections π onto the kernels.

We know already that $\operatorname{tr}_S e^{-tD^2}$ is constant in t! The information for $t \to 0$ is also interesting. From (30) and (31) we have an asymptotic expansion for $t \to 0$

$$\operatorname{tr}_S e^{-tD^2} = \int_M \operatorname{tr}_S k_t(p,p) d\mathrm{vol}(p) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} t^j \int_M \operatorname{tr}_S \Theta_j(p,p) d\mathrm{vol}(p).$$

Note that by construction of the asymptotic development of the heat kernel in Section 5.3 (by iteratively solving ODE's for the functions u_j) each of the endomorphisms $\Theta_j(p,p) : S_p \to S_p$ respects the decomposition into positive and negative parts. Also recall that this integrand is determined by the local geometry of M around p. Since $\operatorname{tr}_S e^{-tD^2}$ is constant in t, we get (n even) that $\operatorname{ind}(D)$ is simply the constant term $\frac{1}{(4\pi)^{n/2}} \int_M \operatorname{tr}_S \Theta_{n/2}(p,p) d\mathrm{vol}(p)$ in the Taylor expansion.

Theorem 141. *[Atiyah-Singer Index Theorem, First Version] For n odd, we have* $\operatorname{ind}(D) = 0$. *For n even,*

$$\operatorname{ind}(D) = \frac{1}{(4\pi)^{n/2}} \int_M \operatorname{tr}_S \Theta_{n/2}(p,p) d\mathrm{vol}(p). \tag{36}$$

It remains to compute the integral on the right. We know that $\Theta_{n/2}(p) = \Theta_{n/2}(p,p)$ only depends on g, ∇^S around p, but their explicit computation is difficult. They will be treated later using Getzler calculus.

Corollary 142 (Homotopy Invariance)**.** *Let $(g_t)_{t\in[0,1]}$ be a smooth family of Riemannian metrics on M. Let $(S_t, \nabla_t) \to M$ be a corresponding family of $\mathbb{Z}/2$-graded Dirac bundles. Then* $\operatorname{ind}(D_0) = \operatorname{ind}(D_1)$.

Proof. The integral in (36) depends continuously on t and is also the integer $\operatorname{ind}(D_t)$. Hence it is constant. □

Remark 143. *This homotopy invariance of the index was observed by Gel'fand in 1959. He then heuristically argued that the index should only depend on topological data (and in particular is independent of the metric g). The explicit computation of the index was called the 'index problem' and was solved by Atiyah-Singer.*

Corollary 144 (Multiplicity for Coverings)**.** *Let $S \to M$ be a Dirac bundle and suppose $\pi\colon M' \to M$ is a d-sheeted covering. Then $S' = \pi^* S \to M'$ is again a Dirac bundle. The index of the Dirac operator D' on S' is then given by*

$$\operatorname{ind}(D') = d \cdot \operatorname{ind}(D)$$

Example 145. *For $n = 2$ let (M^2, g) be a closed oriented Riemann surface. Let D be the Euler operator. Recall that $S_+ = \Lambda^{ev}(T^*M) = \underline{\mathbb{R}} \oplus \underline{\mathbb{R}}$ and $S_- = \Lambda^{odd}(M) = \Lambda^1(T^*M)$. Hence*

$$d + d^* = D_+ \colon C^\infty(M) \oplus C^\infty(M) \to \Omega^1(M).$$

By (36),

$$\chi(M) = \operatorname{ind}(D) = \frac{1}{4\pi} \int_M \operatorname{tr}_S \Theta_1(p) d\mathrm{vol}(p)$$

for the operator $\Theta_1(p)\colon \Lambda^* T_p^* M \to \Lambda^* T_p^* M$ whose restriction to $\Lambda^j T_p^* M \to \Lambda^j T_p^* M$ we denote by $\Theta_1^j(p)$. The right hand side is

$$\frac{1}{4\pi} \int_M \left(\operatorname{tr} \Theta_1^0 - \operatorname{tr} \Theta_1^1 + \operatorname{tr} \Theta_1^2\right) d\mathrm{vol}(p)$$

Identifying $\Omega^2(M) \cong C^\infty(M)$, *the Laplace-Beltrami* $D^2 = dd^* + d^*d\colon C^\infty(M) \oplus C^\infty(M) \to C^\infty(M) \oplus C^\infty(M)$ *restricts to the Laplace operator on functions. Recall from Example* 132 *that*

$$\Theta_1^0(p) = \frac{1}{6}\mathrm{scal}_g(p) = \Theta_1^2(p), \quad \Theta_1^1(p) = \frac{1}{6}\mathrm{scal}_g(p) - \mathrm{Rc}(p)$$

The traces of $\Theta_1^0(p) = \Theta_1^2(p) = \frac{1}{6}\mathrm{scal}_g(p)$ *agree, while* (*note that* S_+ *has rank two*)

$$\mathrm{tr}\,\Theta_1^1(p) = \frac{1}{3}\mathrm{scal}_g(p) - \mathrm{scal}_g(p).$$

Hence we obtain the Gauß-Bonnet Theorem

$$\chi(M) = \frac{1}{4\pi}\int_M \frac{1}{3}\mathrm{scal}_g(p) - \frac{1}{3}\mathrm{scal}_g(p) + \mathrm{scal}_g(p) d\mathrm{vol}(p) = \frac{1}{4\pi}\int_M \mathrm{scal}_g(p) d\mathrm{vol}(p),$$

which therefore turns out to be a special case of the Atiyah-Singer Index Theorem.

6.2 The Getzler Filtration

We return now to the general theory, where we are given a complex $\mathbb{Z}/2$-graded Dirac bundle $S_\pm \to M^{n=2m}$ and corresponding Dirac operator $D_\pm\colon C^\infty(S_\pm) \to C^\infty(S_\mp)$. To make (36) more useful, it remains to understand $\Theta_{n/2}$ and the right hand side in more detail. This will enable us to prove the following main theorem of this course:

Theorem 146 (Atiyah-Singer Index Theorem)**.** *Assume that the grading operator* ε *on* S *is the* canonical grading, *given by multiplication with* $i^m\omega$ *(see the next section). Then we may calculate the index as*

$$\mathrm{ind}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}(S/\Delta).$$

Here, $\hat{A}(TM)$ *and* $\mathrm{ch}(S/\Delta)$ *are differential forms that depend only on the curvature of* M *and* S*, respectively* (*defined below*)*. Via Chern-Weil theory the integral expression on the right can be expressed in terms of characteristic classes of* TM *and* S*.*

For more general gradings on S*, a similar formula holds* (*see Exercise sheet* 14)*.*

Recall that Θ_j was defined in (26) as the asymptotic correction terms for the heat kernel on M, as compared to the heat kernel on the torus. Keeping q fixed, Θ_j can be computed recursively via solutions $u_j = \Theta_j(-, q)$ of the ordinary differential equation

$$\nabla_{\partial/\partial r}(r^j g^{1/4} u_j) = -r^{j-1} g^{1/4} D^2 u_{j-1}, \qquad u_{-1} = 0. \tag{37}$$

(see the proof of Lemma 130.)

6.2.1 Clifford Representations

The following decomposition of an endomorphism is crucial for the Getzler calculus.

Lemma 147. *Let* W *be a complex representation of* $\mathrm{Cl}(n)$*. Then*

$$\mathrm{End}_{\mathbb{C}}(W) = \mathrm{Cl}(n) \otimes \mathrm{End}_{\mathrm{Cl}}(W). \tag{38}$$

Proof. Recall from (57) that the complexified Clifford algebra $\mathbb{C}\mathrm{l}(n) \cong \mathbb{C}^{2^m \times 2^m}$ is a matrix algebra and therefore has a unique irreducible representation $\Delta \cong \mathbb{C}^{2^m}$. If W is another faithful complex $\mathrm{Cl}(n)$-representation W we may therefore write $W = \Delta \otimes_{\mathbb{C}} V$ (so we take $\dim V$ many copies of Δ), where

$$W/\Delta := V = \mathrm{Hom}_{\mathrm{Cl}(n)}(\Delta, W).$$

This follows from Schur's Lemma,

$$\operatorname{Hom}_{\mathrm{Cl}(n)}(\Delta,\Delta)\cong\mathbb{C} \tag{39}$$

It is a special case of the *isotypical decomposition* of a representation. From this we get

$$\operatorname{End}_{\mathbb{C}}(W)=\operatorname{End}_{\mathbb{C}}(\Delta)\otimes_{\mathbb{C}}\operatorname{End}_{\mathbb{C}}(V)=\mathbb{C}\mathrm{l}(n)\otimes_{\mathbb{C}}\operatorname{End}_{\mathbb{C}}(V)\cong\mathrm{Cl}(n)\otimes_{\mathbb{R}}\operatorname{End}_{\mathbb{C}}(V). \tag{40}$$

Similarly, $\operatorname{End}_{\mathrm{Cl}(n)}(W)\cong\operatorname{End}_{\mathrm{Cl}(n)}(\Delta)\otimes\operatorname{End}_{\mathbb{C}}(V)=\operatorname{End}_{\mathbb{C}}(V)$ by (39). □

The next proposition is also fundamental for the proof of the Atiyah-Singer Index Theorem. It implies that in evaluating the integral (36), we need only concentrate on those parts with highest "Clifford degree".

Proposition 148. *Let W be a complex Clifford representation of $\mathrm{Cl}(n)$ for n even, let $F=c\otimes f\in\operatorname{End}_{\mathbb{C}}(W)\cong\mathrm{Cl}(n)\otimes\operatorname{End}_{\mathbb{C}}(V)$ (using the above decomposition), and let $c=\sum c_Ie_I$. Then*

$$\operatorname{tr}_S F=(-2i)^{n/2}c_{1,\dots,n}\operatorname{tr}f=:(-2i)^{n/2}c_{1,\dots,n}\operatorname{tr}^{W/\Delta}(F). \tag{41}$$

(*Using the canonical isomorphism $W=\Delta\otimes_{\mathbb{C}}V$, the $\mathbb{Z}/2$-grading defines a grading on W. The super-trace of the endomorphism F of W is with respect to this grading.*)

Proof. Viewing elements of $\mathbb{C}\mathrm{l}(n)=\operatorname{End}_{\mathbb{C}}(\Delta)$ as endomorphisms of Δ, we have[4]

$$\operatorname{tr}F=\operatorname{tr}c\cdot\operatorname{tr}f. \tag{42}$$

On Δ we have a *canonical* $\mathbb{Z}/2$-grading given by $\varepsilon=i^m\omega$ for the volume element $\omega=e_1\cdots e_n$. Then

$$W=(\Delta_+\otimes V)\oplus(\Delta_-\otimes V)$$

defines $\mathbb{Z}/2$-grading on W. Similar to (42) we then find for the super trace of $F\in\operatorname{End}_{\mathbb{C}}(W)$

$$\operatorname{tr}_S F=\operatorname{tr}_S c\cdot\operatorname{tr}f.$$

It remains only to prove the following lemma. □

Lemma 149. *Let $c=\sum_{I\subset\{1,\dots,n\}}c_I\cdot e_I\in\mathrm{Cl}(n)$ where $e_I=\prod_{i\in I}e_i$ for the standard basis e_i of $\mathbb{R}^{n=2m}$ and $c_I\in\mathbb{R}$. Then, regarding $c\in\operatorname{End}(\Delta)$, we have*

$$\operatorname{tr}_S(c)=(-2i)^m c_{\{1,\dots,n\}}.\qquad\textit{(coefficient of the top part of c)}$$

Proof. For $c=e_I$ we have

$$\operatorname{tr}_S(c)=\operatorname{tr}(\varepsilon\circ c)=\operatorname{tr}(i^m\omega c)$$

and

$$i^m\omega e_I=\begin{cases}(-1)^m i^m & \text{for } I=\{1,\dots,n\} \qquad \text{(recall } \omega^2=(-1)^m\text{)}\\ \lambda e_J & \text{for } I\neq\{1,\dots,n\} \qquad \text{(for some } \lambda\in\mathbb{C} \text{ and } J\neq\emptyset \text{ complementary to } J\text{)}\end{cases}$$

Since $e_\emptyset=\mathrm{id}\colon\Delta\to\Delta$, we have $\operatorname{tr}(e_\emptyset)=2^m$. However, for $I\neq\emptyset$ we get $\operatorname{tr}(e_I)=0$. This is because $\operatorname{End}(\Delta)=\Delta\otimes\Delta^*$ and e_I acts on both Δ and $\operatorname{End}(\Delta)$ by left multiplication on Δ. Let us denote the action of e_I on $\operatorname{End}(\Delta)$ by ε_I. Then, as $\dim\Delta^*=2^m$,

$$\operatorname{tr}(e_I)=\frac{1}{2^m}\operatorname{tr}(\varepsilon_I).$$

But ε_I is left-multiplication on $\mathbb{C}\mathrm{l}(n)=\operatorname{End}(\Delta)$ and permutes the basis $e_{i_1}\cdots e_{i_k}$ (up to scalars) without fixing any basis vector as $I\neq\emptyset$. Hence it has trace equal to zero. □

[4] More generally, $\operatorname{tr}(A\otimes B)=\operatorname{tr}(A)\cdot\operatorname{tr}(B)$ for endomorphisms $A\in\operatorname{End}(V)$, $B\in\operatorname{End}(W)$ of finite-dimensional vector spaces

6.2.2 Taylor Expansion

Let V be a finite-dimensional vector space. Recall that $\mathbb{C}[\![V]\!]$ is the $\mathbb{C}$-algebra of formal power series in elements of V and coefficients in $\mathbb{C}$. More formally, $\mathbb{C}[\![V]\!]$ is the quotient of the tensor algebra TV by the ideal generated by $v \otimes w - w \otimes v$ for $v, w \in V$. It is the free commutative $\mathbb{C}$-algebra generated by V.

Definition 150. *Let $s \in C^\infty(S \boxtimes S^*)$. Fixing q we get a section $s_q = s(-,q)$ of the bundle $S \otimes S_q^*$. For p close to q we may use parallel transport along the geodesic from q to p to identify this bundle with the trivial bundle* $\mathrm{End}(S_q)$. *Precomposing with the exponential map gives (defined in a neighborhood of zero)*

$$T_qM \xrightarrow{\exp} M \xrightarrow{s_q} \mathrm{End}(S_q)$$

which is a smooth map of finite-dimensional vector spaces. The Taylor expansion *of this map is an element of $\mathbb{C}[\![T_q^*M]\!] \otimes \mathrm{End}(S_q)$. In more detail, if we choose normal coordinates (x^i) centered at q (so choosing an orthonormal basis of T_xM), this is just the ordinary Taylor expansion of a vector-valued map $S_q \colon \mathbb{R}^n \to$* $\mathrm{End}(S_q)$. *We write this* asymptotic expansion *as*

$$s_q \sim \sum_\alpha s_\alpha x^\alpha, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \;\; s_\alpha \in \mathrm{End}(S_q) \tag{43}$$

for

$$s_\alpha = \frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{|\alpha|} S_q}{\partial x^\alpha}(0).$$

Letting q vary, Taylor expansion clearly defines a section $\tau(s)$ of $\mathbb{C}[\![T^*M]\!] \otimes \mathrm{End}(S)$.

Remark 151.

1. *We have used the short-hand notation $x^i = dx^i$. Then the right-hand side of* (43) *defines an element of $\mathbb{C}[\![T_q^*M]\!] \otimes \mathrm{End}(S_q)$.*
2. *$s(q,q)$ is the constant term $\tau(s)(q)_0$ of the Taylor expansion.*
3. *As familiar from calculus, the Taylor expansion needn't converge or represent our function.*
4. *Working in arbitrary coordinates, higher derivatives of functions are usually not well-defined (leading to jet bundles). In the context of Riemannian manifolds, we may use normal coordinates at q which are well-defined up to a linear transformation. Then higher derivatives are well-defined.*

As an endomorphism of S_q it can be written in the form $s_0 = \sum_\mu c_\mu \otimes f_\mu \in \mathrm{End}(S_q)$ as in Proposition 41 so $\mathrm{tr}_S\, s_0 = \sum_\mu \mathrm{tr}_S\,(c_\mu) \cdot \mathrm{tr}\,(f_\mu)$, where $\mathrm{tr}_S\, c_\mu$ was computed by looking at the coefficient of the top Clifford part of $c_\mu \in \mathrm{Cl}(T_qM)$ (at least if S_q carries the canonical grading).

Plan: Develop a calculus that assigns simpler objects to sections in $C^\infty(S \boxtimes S^*)$ (viewed as smoothing kernels of smoothing operators $C^\infty(S) \to C^\infty(S)$) and also to differential operators $C^\infty(S) \to C^\infty(S)$. These simpler objects should have a 'degree'. Clifford multiplication with $v \in T_qM$ should raise the degree by 1, while multiplication with a coordinate function x^i should lower the degree by 1 (in a sense to be made precise). To compute the trace it suffices then to consider the contribution in top degree $n = \dim M$. This assignment should be compatible with the composition of differential operators with smoothing operators: For a differential operator $P \in \mathcal{D}(S)$ and a smoothing operator $Q \colon C^\infty(S) \to C^\infty(S)$ with smoothing kernel s, i.e.

$$Q(\varphi)(p) = \int_M s(p,q)\varphi(q)dq,$$

recall that the composite $P \circ Q$ is the smoothing operator with kernel $P_p s(\cdot,\cdot)$ where P_p denotes the differential operator P acting only on the p-entry of s.

6.2.3 Filtered Algebras and Symbol Maps

We start with the differential operators. Recall that $\mathcal{D}(S)$ is the algebra of differential operators $C^\infty(S) \to C^\infty(S)$. Recall our traditional filtration

$$\mathcal{D}(S) \supset \cdots \supset \mathcal{D}_k(S) \supset \mathcal{D}_{k-1}(S) \supset \cdots \supset \mathcal{D}_0(S) = \mathrm{End}(S).$$

by differential operators $\mathcal{D}_k(S)$ of order up to k.

We recall the following general definition:

Definition 152. *A* filtered algebra *is an algebra A with a sequence of vector subspaces A_k satisfying*

$$A_k \supset A_{k-1}, \qquad A_k \cdot A_l \subset A_{k+l}, \qquad \bigcup A_k = A.$$

On every filtered algebra we have the *order function* $f\colon A \to \mathbb{Z} \cup \{-\infty\}, f(x) = \min\{k \in \mathbb{Z} \mid x \in A_k\}$. The order function satisfies

$$f(x+y) \le \max(f(x), f(y)), \quad f(x \cdot y) \le f(x) + f(y), \quad f(0) = -\infty, \quad f(\lambda x) \le f(x)$$

for $x, y \in A$ and $\lambda \in \mathbb{C}$. Conversely, given a function $f\colon A \to \mathbb{Z} \cup \{-\infty\}$ on A with these properties, we may define a filtration by

$$A_k = \{x \in A \mid f(x) \le k\}.$$

We shall also write $A_{\le k}$ for A_k. Thus, order functions and filtrations are equivalent points of view. For us, the main example of a filtered algebra will be the differential operators $A = \mathcal{D}(S)$.

Definition 153. *1. The* trivial filtration *on V is given by $V_k = V$ for all k.*

2. Let $V_k \supset V_{k-1}$ and $W_l \supset W_{l-1}$ be filtrations of V and W. Then $(V \otimes W)_n = \bigoplus_{k+l=n} V_k \otimes W_l$ defines the tensor product filtration *of $V \otimes W$.*

3. If $f\colon W \to V$ is a homomorphism and V is filtered, then $W_k = f^{-1}(V_k)$ defines the pullback filtration *on W.*

Definition 154. *A graded algebra is an algebra A with a decomposition $A = \bigoplus A^m$ into vector subspaces satisfying $A^n \cdot A^m \subset A^{n+m}$ for the product.*

We take it as a convention that upper indices denote a grading, while lower indices denote a filtration. Any graded algebra may be regarded as filtered, by $A_{\le k} = \bigoplus_{i \le k} A^i$. There is also an obvious grading on the tensor product of graded algebras. We call an algebra trivially graded if $A = A^0$ is only in degree zero.

Definition 155. *Let A_m be a filtered algebra. A symbol map consists of a graded $\mathbb{C}$-algebra $\mathcal{G}^* = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \cdots$ and a family of linear maps $\sigma_m\colon A_m \to \mathcal{G}^m$ with the following properties:*

1. $\sigma_m|_{A_{m-1}} = 0$

2. $\sigma_m(a) \cdot \sigma_{m'}(a') = \sigma_{m+m'}(aa')$ for $a \in A_m$, $a' \in A_{m'}$

Example 156. *1. Let $\mathcal{G}^m = A_m/A_{m-1}$ with the projection $\sigma_m\colon A_m \to \mathcal{G}^m$, the* universal symbol. *The graded algebra $\bigoplus_{m \ge 0} \mathcal{G}^m$ is called the* associated graded algebra *of the filtered algebra A.*

2. $A = \mathrm{Cl}(n)$ and $A_m = \mathrm{Span}\{e_I \mid |I| \le m\}$. The associated graded algebra $\mathcal{G}^m = A_m/A_{m-1}$ is isomorphic to the exterior algebra $\mathcal{G}^* = \Lambda^* \mathbb{R}^n$. Here, for example, σ_2 maps the Clifford product $v \cdot w$ to $v \wedge w$, compare Proposition 22.

6.2.4 The Principal Symbol

Definition 157. *For vector bundles $E, F \to M$ let $P \in \mathcal{D}_m(E, F)$ be a differential operator of order $\leq m$. Let $\xi \in T_x^*M$ be a cotangent vector and let $f\colon M \to \mathbb{R}$ be a smooth function with $f(x) = 0$ and $df(x) = \xi$. We define the* principal symbol *of P with respect to ξ as the linear map $\sigma_m(P, \xi) : E_x \to F_x$ given by*

$$\sigma_m(P, \xi)(e) := \frac{1}{m!} P(f^m \tilde{e})_x \in F_x, \qquad e \in E_x.$$

Here $\tilde{e} \in C^\infty(E)$ is an extension of e to a smooth section.

We need to show that this is well-defined. For this, let $(x^1, \ldots, x^n)$ be local coordinates near x and write $\xi = \xi_i dx^i \in T_x^*M$. Write $P = \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}$. Then

$$\begin{aligned}
\frac{1}{m!} P(f^m \tilde{e})_x &= \frac{1}{m!} \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}(f^m \tilde{e})}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}(x) \\
&\overset{f(x)=0}{=} \frac{1}{m!} \sum_{|\alpha| = m} A^\alpha(x) \frac{\partial^{|\alpha|}(f^m)}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}(x) \cdot e \\
&= \sum_{|\alpha|=m} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} A^\alpha(x) \cdot e
\end{aligned}$$

Obviously, the right-hand side is independent of $\tilde{e}$ and f. On the other hand, our definition is independent of the representation of the differential operator P in local coordinates. Hence the symbol is independent of both.

Remark 158. *Let us think about the variance of this expression under coordinate change. From the coordinates $(x^1, \ldots, x^n)$ we get a frame $(\partial_1, \ldots, \partial_n)$ of T_xM whose coordinates $(\eta^1, \ldots, \eta^n)$, viewed as linear maps $T_xM \to \mathbb{R}$, are equal to the cotangent vectors $\eta^i = dx^i$. Similarly, the frame $(dx^1, \ldots, dx^n)$ of T_x^*M leads to coordinates $(\xi_1, \ldots, \xi_n)$ which, when viewed as linear maps $T_x^*M \to \mathbb{R}$, may be identified with $\xi_i = \partial_i$ (by identifying the double dual of a vector space with the vector space itself). Hence the entries ξ_i in the above expression for the symbol, viewed as linear maps $T_x^*M \to \mathbb{R}$, $\xi \mapsto \xi_i(\xi)$, can be identified with $\partial_i \in T_xM$.*

Hence, identifying $\xi_i = \partial_i$, we may view the *m-th principal symbol map* as a section (cf. [Roe])

$$\sigma(P) \in C^\infty(\mathbb{C}[TM] \otimes \mathrm{Hom}(E, F)), \qquad \sigma(P)(e) := \sum_{|\alpha|=m} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} A^\alpha \cdot e,$$

where $\mathbb{C}[V] = \bigoplus_{m \geq 0} \mathrm{Sym}^m(V)$, where $\mathrm{Sym}^m(V) \subset V^{\otimes m}$ are the symmetric tensors (meaning that they are invariant under the obvious action of the symmetric group Σ_m on $V^{\otimes m}$), or in other words, formal homogenous polynomials in elements of V of degree m.

Example 159. *Consider $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$, so $m = 1$. Let $\xi = \xi_i dx^i \in T_x^*M$. For $\omega \in \Lambda^k T_x^*M$ extended to a k-form on M we have*

$$\sigma(d, \xi)\omega = d(f\omega)|_x = (df \wedge \omega + f d\omega)|_x \overset{f(x)=0}{=} \xi \wedge \omega.$$

6.2.5 The Getzler Filtration

Let $S \to M$ be a complex Dirac bundle. From Equation 40 we have an isomorphism

$$\mathrm{End}_{\mathbb{C}}(S) = \mathrm{Cl}(TM) \otimes \mathrm{End}_{\mathrm{Cl}(TM)}(S),$$

which will be used to define a filtration on $\mathrm{End}_{\mathbb{C}}(S)$.

Recall from Proposition 20 that the Clifford algebra is filtered by $\mathrm{Cl}^{(k)}(V) = \pi(\bigoplus_{r=0}^k V^{\otimes r})$, for the canonical projection $\pi\colon TV \to \mathrm{Cl}(V)$. Equivalently, $\mathrm{Cl}^{(k)}(V) = \lambda(\Lambda^k V)$ using the isomorphism $\lambda\colon \Lambda^*(V) \to \mathrm{Cl}(V)$ of Proposition 20. This is called the *Clifford filtration.*

Definition 160. *The* Clifford filtration *on* $\mathrm{End}_{\mathbb{C}}(S)$ *is the tensor product of the Clifford filtration of* $\mathrm{Cl}(TM)$ *and the trivial filtration on* $\mathrm{End}_{\mathrm{Cl}(TM)}(S)$.

Utilizing this filtration for differential operators $\mathcal{D}(S)$, we will define the Getzler filtration on differential operators. By Definition 59 and (40), the differential operators $\mathcal{D}(S)$ are generated as an algebra by

1. $\mathrm{End}_{\mathrm{Cl}(TM)}(S)$,
2. $X \in C^\infty(TM)$ (we write $c(X) \in \mathrm{End}(S)$ for the corresponding Clifford multiplication),
3. ∇_X for $X \in C^\infty(TM)$.

We define filtration by specifying the order function on these generators:

Definition 161. *The* Getzler filtration *on* $\mathcal{D}(S)$ *is defined by*

1. *For* $\varphi \in \mathrm{End}_{\mathrm{Cl}(TM)}(S)$ *we let* $\mathrm{ord}^{\mathcal{G}}(\varphi) = 0$.
2. *For* $X \in C^\infty(TM)$ *Clifford multiplication* $c(X)$ *has* $\mathrm{ord}^{\mathcal{G}}(c(X)) = 1$.
3. *For* $X \in C^\infty(TM)$ *let* $\mathrm{ord}^{\mathcal{G}}(\nabla_X) = 1$.

In other words, the Getzler filtration *is*

$$\mathcal{D}^{\mathcal{G}}_m(S) = \mathrm{span}_{\mathbb{C}} \left\{ P = \lambda_1 \cdots \lambda_k \mid \sum_{i=1}^{k} \mathrm{ord}^{\mathcal{G}} \lambda_i \leq m \right\}$$

where $\lambda_1, \ldots, \lambda_k$ *are generators as in 1., 2., 3. We write* $\mathcal{D}^{\mathcal{G}}(S)$ *for the algebra of differential operators with this filtration.*

Note that convention 2. is different from our old filtration convention (where the order was zero). It is our goal to define the *Getzler symbol* map

$$\sigma_* \colon \mathcal{D}^{\mathcal{G}}(S) \to C^\infty\left(\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S)\right). \tag{44}$$

Definition 162. *Define a vector bundle over* M *by*

$$\mathfrak{P}(TM) := \mathbb{C}[T^*M] \otimes \mathbb{C}[TM].$$

At $x \in M$, $\varphi \in \mathfrak{P}(TM)_x$ *may be written as finite sum*

$$\varphi = \sum \mu_I^J dx^I \otimes \partial_J, \quad \mu_I^J \in \mathbb{C}.$$

We will regard φ *as a differential operator on smooth functions on* T_xM

$$\varphi(x) \colon C^\infty(T_xM, \mathbb{C}) \to C^\infty(T_xM, \mathbb{C}), \quad f \mapsto \sum \varphi_I^J x^I \frac{\partial^{|J|} f}{\partial x^J}(x)$$

with polynomial coefficients $\varphi^J = \sum_I \varphi_I^J \in \mathbb{C}[T_x^*M]$ *in the coordinates* $x^i := dx^i \colon T_xM \to \mathbb{R}$ *on* T_x^*M. *In particular, the algebra structure on* $\mathfrak{P}(TM)$ *is* not *the tensor product of algebras. Instead, we have*

$$(x^I \partial_J) \cdot \left(\sum x^K \partial_L\right) = x^I \sum_{\alpha+\beta=J} \binom{\alpha}{\beta} \partial_\alpha(x^K) \partial_{\beta+L}.$$

Note that in [Roe] the coordinate $dx^i : T_xM \to \mathbb{R}$ is denoted x^i again. This is justified by the fact that under the canonical identification $T_0(T_xM) = T_xM$ this coordinate satisfies the tautological equation $\frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^i}$. This amounts to the intuition that the linear map $x^i = dx^i : T_xM \to \mathbb{R}$ is the infinitesimal version of the coordinate $x^i : M \to \mathbb{R}$. We will follow this convention.

Remark 163. *For $\varphi \in C^\infty(\mathbb{C}[TM]) \subset C^\infty(\mathbb{C}[T^*M] \otimes \mathbb{C}[TM])$ we have $\varphi(x) = \sum \mu^J \partial_J$ with $\mu^J \in \mathbb{C}$, which gives a differential operator $C^\infty(T_xM, \mathbb{R}) \to C^\infty(T_xM, \mathbb{R})$ for each $x \in M$ with* constant (*as opposed to polynomial*) *coefficients. In accordance with* [Roe], *we also write $\mathfrak{C}(TM) = \mathbb{C}[TM]$. We then have*

$$C^\infty(\mathfrak{C}(TM) \otimes \mathrm{Cl}(TM) \otimes \mathrm{End}_{\mathrm{Cl}}(S)) = C^\infty(\mathfrak{C}(TM) \otimes \mathrm{End}_{\mathbb{C}}(S))$$

which is the target of the principal symbol map on $\mathcal{D}(S)$, compare Example 12.8 *in* [Roe]. *Hence, for the Getzler symbol map we replace $\mathfrak{C}[TM]$ by the refined $\mathfrak{P}[TM]$ and the bundle* $\mathrm{Cl}(TM)$ (*viewed as a filtered algebra in the Getzler calculus*) *by the associated graded algebra Λ^*TM.*

According to our Definition 155 of symbol maps, we must define a grading on the target bundle of algebras $\mathfrak{P}(TM) \otimes \Lambda^*T^*M \otimes \mathrm{End}_{\mathrm{Cl}}(S)$ of the Getzler symbol (44). We will use the tensor product grading, where $\Lambda^*(T^*M)$ gets the usual grading and $\mathrm{End}_{\mathrm{Cl}}(S)$ has the trivial grading. Finally, on $\mathfrak{P}(TM)$ we define:

Definition 164. *Homogeneous elements $\varphi(x) = x^I \otimes \partial_J \in \mathfrak{P}(T_xM)$ are defined to have degree $|J| - |I|$.*

$$\mathfrak{P}(T_xM)^k = \left\{ \sum_{|J|-|I|=k} \mu_I^J x^I \otimes \partial_J \;\middle|\; \mu_I^J \in \mathbb{C} \right\}$$

As a final preparation, we consider the following section, defined for each vector field $X \in C^\infty(TM)$

$$(RX, \cdot) \in C^\infty(T^*M \otimes \Lambda^2T^*M) \subset C^\infty(\mathfrak{P}(TM) \otimes \Lambda^2T^*M) \tag{45}$$

of degree 1. At $x \in M$ it is the map $T_xM \to \Lambda^2T_x^*M$ given by $(RX(-,-), Y)$. Stated otherwise,

$$(RX, \cdot) \in C^\infty(\Lambda^2T^*M \otimes T^*M), \quad (S \wedge T) \otimes Y \mapsto (R(S,T)X, Y).$$

Recall that the Riemannian curvature tensor $R \in \Omega^2(M; \mathrm{End}(TM))$ can locally be expressed by functions $R_{ijkl} = -g(R(e_i, e_j)e_k, e_l) \in C^\infty(M)$ for any local (not necessarily orthonormal) frame $(e_1, \dots, e_n)$ of TM. The R_{ijkl} are antisymmetric for the first two and the last two entries and symmetric with respect to interchanging i, j with k, l, see Equation (5). Let us work out (45) in local coordinates $(x^1, \dots, x^n)$ on M with $e_i = \partial_{x_i}$. We define $(R_{ij})_{i,j=1}^n$ to be the $(n \times n)$-matrix of 2-forms

$$R_{ij} = \sum_{k<l} R_{ijkl} dx^k \wedge dx^l = R_{ij} = \sum_{k<l} R_{klij} dx^k \wedge dx^l = -(R\partial_i, \partial_j) \in \Lambda^2T^*M.$$

(using $R_{ijkl} = R_{klij}$.) The local expression of $(RX, \cdot)$ for $X = a^i\partial_i$ used in [Roe] and in the literature is

$$R(X, \cdot) = -a^i R_{ij} x^j.$$

In particular, the degree of (45) is one.

6.3 The Getzler Symbol

Recall that the bundle of algebras $\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S)$ is graded by the tensor product grading. As explained before, the degree of $x^I\partial_J \in \mathfrak{P}(TM)$ is $|J| - |I|$, we take the usual grading on differential forms, and let every element of $\mathrm{End}_{\mathrm{Cl}}(S)$ have degree zero. Note that we also get non-zero elements in negative gradings. The sections of this bundle is then a graded algebra, which a possible target for a symbol map in the sense of Definition 155.

6.3.1 Statement of the Main Theorem

We will use the obvious inclusions

$$TM, T^*M \hookrightarrow \mathfrak{P}(TM), \qquad \mathfrak{P}(TM), \Lambda^*(T^*M), \mathrm{End}_{\mathrm{Cl}}(S) \hookrightarrow \mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S).$$

Theorem 165. *There exists a symbol map (the* Getzler symbol*)*

$$\sigma_m^{\mathcal{G}} \colon \mathcal{D}_m^{\mathcal{G}}(S) \to C^\infty\left(\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S)\right)^m$$

uniquely characterized by the properties (see Definition 161 *for definition of the Getzler filtration on $\mathcal{D}^{\mathcal{G}}(S)$).*

1. $\sigma_0^{\mathcal{G}}(\varphi) = \varphi$ *for* $\varphi \in \mathrm{End}_{\mathrm{Cl}}(S)$*.*
2. $\sigma_1^{\mathcal{G}}(c(X)) = X^\flat \in \Lambda^*(T^*M)$ *for* $X \in C^\infty(TM)$*.*
3. $\sigma_1^{\mathcal{G}}(\nabla_X) = \partial_X - \frac{1}{4}(RX, \cdot) \in C^\infty(\mathfrak{P}(TM) \otimes \Lambda^2(T^*M))$ *for* $X \in C^\infty(TM)$*. (Note that $\partial_X := X$ can be canonically viewed as a section in $C^\infty(\mathfrak{P}(TM)$.)*

Proof. Uniqueness is clear, since we have defined the symbol on all generators of the algebra $\mathcal{D}(S)$ of differential operators. The problem with existence is that a differential operator may be decomposed in several ways into the generators; it is not clear that our definition will be coherent. We will prove this by considering the action of differential operators $T \in \mathcal{D}(S)$ on smoothing operators Q by post-composition $T \circ Q$ (whose kernel $T\left(k_Q(\cdot, q)\right)$ is obtained by applying T only in p-direction). This will require some preparation, in particular we must define the symbol of a smoothing operator. This is done in the next section, after which we return to the proof of Theorem 165. □

6.3.2 The Canonical Symbol on Smoothing Operators

We now define the symbol of a smoothing operator. For clarity, we shall call it the *canonical symbol*, although it may also be viewed as a generalization of the Getzler symbol to pseudo-differential operators. We begin by defining the *canonical filtration* on smoothing operators $C^\infty(S \boxtimes S^*)$.

Fix geodesic normal coordinates $(x^1, \dots, x^n)$ near a point $q \in M$. The Taylor expansion of the smoothing kernel $s = k_Q(-, q)$ near q has the form (see Definition 150)

$$s \sim \sum s_\alpha x^\alpha$$

and may be regarded as a section of $\mathbb{C}[\![T_q^*M]\!] \otimes \mathrm{End}(S_q) \cong \mathbb{C}[\![T_q^*M]\!] \otimes \mathrm{Cl}(T_qM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_q)$. For varying q we hence obtain the *Taylor expansion map*

$$\tau \colon \{Q : C^\infty(S) \to C^\infty(S) \mid Q \text{ smoothing operator}\} \to C^\infty(\mathbb{C}[\![T^*M]\!] \otimes \mathrm{End}(S)).$$

The power series algebra $\mathbb{C}[\![T^*M]\!]$ has a filtration by assigning to $x^\alpha (= (dx)^\alpha)$ the filtration degree $-|\alpha|$, as usual. Recall the Clifford filtration on $\mathrm{End}(S)$ from Definition 160. Passing to the associated graded algebra defines the *Clifford symbol*

$$\sigma_k^{\mathrm{Cl}} \colon \mathrm{End}(S)_{\leq k} = \mathrm{Cl}^{(k)}(T_qM) \otimes \mathrm{End}_{\mathrm{Cl}}(S) \to \Lambda^k(T_qM) \otimes \mathrm{End}_{\mathrm{Cl}}(S).$$

We take the tensor product filtration on $\mathbb{C}[\![T^*M]\!] \otimes \mathrm{End}(S)$.

Definition 166. *The* canonical filtration *on $C^\infty(S \boxtimes S^*)$ is the pullback filtration along the map τ.*

Hence the filtration degree of a smoothing operator is defined by passing to its Taylor expansion: if at every point $q \in M$ we have $s(-, q) \sim \sum s_\alpha x^\alpha$ (for normal coordinates (x^i) at q) with $s_\alpha \in \mathrm{End}(S_q)_{\leq k+|\alpha|}$ in Clifford filtration, then by definition s has canonical filtration $\leq k$. Notice that in this filtration x^α has order $-|\alpha|$ so that we get elements of filtration order m for all $m \in \mathbb{Z}$.

Definition 167. *Suppose $s \in C^\infty(S \boxtimes S^*)$ lies in canonical filtration $\leq k$. For fixed $q \in M$ let $s(-, q) \sim \sum s_\alpha x^\alpha$ in normal coordinates at q. The* Clifford symbol *$\sigma(s)|_q$ of s at $q \in M$ is*

$$\sum \sigma_{k+|\alpha|}^{\mathrm{Cl}}(s_\alpha) x^\alpha \in \mathbb{C}[\![T_q^*M]\!] \otimes \Lambda^*(T_qM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_q).$$

Letting q vary, identifying $TM \cong T^*M$ by the metric, and projecting to the homogenous part of degree k on the right hand side, we get the canonical symbol

$$\sigma_k \colon C^\infty(S \boxtimes S^*)_{\leq k} \to C^\infty(\mathbb{C}[T^*M] \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))^k.$$

Note that the target of this map has the tensor product grading of the graded algebras $\mathbb{C}[T^*M]$, $\Lambda^*(T^*M)$, and $\mathrm{End}_{\mathrm{Cl}}(S)$, the last being trivially graded (with degree 0). Also note that $\Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S)$ sits only in finitely many degrees so that we can replace the power series algebra $\mathbb{C}[\![T^*M]\!]$ by the polynomial algebra $\mathbb{C}[T^*M]$. This is in contrast to [Roe], paragraph in front of Definition 12.20, who keeps $\mathbb{C}[\![T^*M]\!]$ in the target of the canonical symbol. It seems that with Roe's definition the symbol σ_k does not vanish on smoothing operators of canonical filtration degree $\leq k-1$, which leads to difficulties in the proof of Lemma 168, which corresponds to [Roe], Prop. 12.22.

Our map σ_k (by definition) maps $C^\infty(S \boxtimes S^*)_{\leq k-1}$ to zero. Note, however, that although we can compose smoothing operators and hence put an algebra structure on the set of smoothing operators $C^\infty(S) \to C^\infty(S)$, this map is *not* compatible with σ. Therefore σ is not a symbol map in the sense of Definition 155. Rather we will use the compatibility of the canonical symbol σ with the Getzler symbol σ^G on differential operators under the natural action of differential operators on smoothing operators given by composition, see Lemma 168 below.

6.3.3 Proof of Theorem 165

The bundle $C^\infty(\mathbb{C}[T^*M] \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))$ is a module over $C^\infty(\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))$ by means of the bundle map

$$\mathfrak{P}(TM) \otimes \mathbb{C}[T^*M] \to \mathbb{C}[T^*M]$$

defined at $x \in M$ by letting a differential operator $\mu_I^J x^I \partial_J : C^\infty(T_xM, \mathbb{C}) \to C^\infty(T_xM, \mathbb{C})$ with polynomial coefficients $\mu_I^J x^I \in \mathbb{C}[T_x^*M]$ act on a polynomial $\sum c_\alpha x^\alpha \in \mathbb{C}[T_x^*M]$ as a derivation in the usual way. Note that this is defined by the usual linear action of ∂_i on $x^j = dx^j$ (i.e. $\partial_i(x^j) = \delta_i^j$) and the Leibniz rule on products x^α. Hence we indeed get a bundle map. On $\Lambda^*(T^*M)$ and $\mathrm{End}_{\mathrm{Cl}}(S)$ the module map is induced by the usual multiplication. This module structure is used in the following lemma to compose the symbols.

Lemma 168. *Let $T \in \mathcal{D}^{\mathcal{G}}(S)$ be a generator of the algebra of differential operators, so $\mathrm{ord}^{\mathcal{G}}(T) = m \in \{0, 1\}$. Let Q be a smoothing operator in filtration $\leq k$. Then $T \circ Q$ is a smoothing operator in filtration $\leq k+m$ and the canonical symbols are related by*

$$\sigma_{k+m}(T \circ Q) = \sigma_m^{\mathcal{G}}(T) \cdot \sigma_k(Q).$$

(here $\sigma_m^{\mathcal{G}}(T)$ is the Getzler symbol of the generator T as defined in Theorem 165.*)*

Proof of Theorem 165. Decompose an arbitrary $T \in \mathcal{D}^{\mathcal{G}}(S)_m$ into generators $S_1 \cdots S_r$ of Getzler order ≤ 1 and so that their orders add up to m (by definition of the Getzler filtration) and then an inductive application of our lemma shows that $T \circ Q$ has Getzler order $\leq k+m$ and

$$\sigma_{k+m}^{\mathcal{G}}(TQ) = \sigma^{\mathcal{G}}(S_1) \cdots \sigma^{\mathcal{G}}(S_r) \cdot \sigma_k^{\mathcal{G}}(Q) \in C^\infty(\mathbb{C}[T^*M] otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S)).$$

But $\sigma_k^{\mathcal{G}}(Q)$ and $\sigma_{k+m}^{\mathcal{G}}(TQ)$ are defined without any reference to the decomposition of T, so the algebra element $\sigma^{\mathcal{G}}(S_1) \cdots \sigma^{\mathcal{G}}(S_r) \in C^\infty(\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))$ is uniquely determined by this equation, because $C^\infty(\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))$ acts faithfully on the module $C^\infty(\mathbb{C}[T^*M] \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))$. $\square$

Definition 169. *A section of a bundle $E \to M$ with connection over a Riemannian manifold M is said to be* synchronous *at $q \in M$ if for all points p in a neighborhood of q the section is parallel along the geodesic from q to p.*

Proof of Lemma 168. Fix $q \in M$ and let (x^i) denote normal coordinates at q. Let $k_Q \in C^\infty(S \boxtimes S^*)_k$ be the smoothing kernel of Q and consider the Taylor expansion

$$s := k_Q(-, q) \sim \sum s_\alpha x^\alpha \in (\mathbb{C}[\![T_q^*M]\!] \otimes \mathrm{Cl}(T_qM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_q))_{\leq k},$$

where $s_\alpha \in \mathrm{End}(S_q)_{\leq k+|\alpha|}$. We consider the generators T case by case.

1. Suppose $\varphi \in C^\infty(\mathrm{End}_{\mathrm{Cl}}(S))$. Consider the Taylor expansion around q

$$\varphi \sim \sum \varphi_\beta x^\beta, \qquad \varphi_\beta \in \mathrm{End}_{\mathrm{Cl}}(S_q).$$

Then $\varphi(q) = \varphi_0$ and

$$\varphi \circ s = \sum_\alpha \underbrace{\varphi_0(s_\alpha)x^\alpha}_{\text{filt } \le k} + \sum_{\alpha, |\beta|>0} \underbrace{\varphi_\beta(s_\alpha)x^{\alpha+\beta}}_{\text{filt } \le k+|\alpha|-|\alpha|-|\beta| \le k-1}$$

lies in canonical filtration $\le k$ and has canonical symbol $\sum_\alpha \sigma^{\mathrm{Cl}}_{k+|\alpha|}(\varphi_0(s_\alpha))x^\alpha = \sigma^{\mathcal{G}}(\varphi)\sigma(s)|_q$.

2. Let $X \in C^\infty(TM)$ and Taylor expand $X \sim \sum X_\beta x^\beta$ around q, where $X_0 = X_q$. Then

$$c(X)s = \sum_\alpha \underbrace{c(X_0)s_\alpha x^\alpha}_{\text{filt } \le 1+k} + \sum_{\alpha,|\beta|>0} \underbrace{c(X_\beta)s_\alpha x^{\alpha+\beta}}_{\text{filt } \le 1+k-|\beta| \le k}$$

has canonical filtration $\le k+1$ with $\sigma(c(X)s) = \sum_\alpha X_q^\flat \wedge \sigma^{\mathrm{Cl}}_{k+|\alpha|}(s_\alpha)x^\alpha = \sigma_1^{\mathcal{G}}(c(X))\sigma(s)|_q$.

3. Let $\partial_i = X \in C^\infty(TM)$ and consider $T = \nabla_X$. We begin by considering the case where s in synchronous at q, so that $s_\alpha = 0$ except for $s_0 = s(q)$. Write

$$\nabla_X s \sim \sum t_\alpha x^\alpha.$$

Let $Y = \sum x^j \partial_j$ be the unnormalized radial vector field. We have $[X,Y] = X = \partial_i$, $\nabla_Y s = 0$, and $Y^\alpha x^\alpha = |\alpha| x^\alpha$. Consider the Taylor expansion of $K^S(X,Y)\cdot s$ for the curvature tensor K^S of S:

$$-\sum(|\alpha|+1)t_\alpha x^\alpha \sim \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s = K^S(X,Y)\cdot s \sim \sum_{j,\alpha}(K^S_{ij})_\alpha s_0 x^j x^\alpha. \qquad (46)$$

There is a decomposition[5] $K^S_{ij} = R^S_{ij} + F^S_{ij}$ into the Riemannian endomorphism $R^S_{ij} \in \mathrm{End}(S_q)_{\le 2}$ with $\sigma_2^{\mathrm{Cl}}(R^S_{ij}) = -\frac{1}{2}R_{ij}$ and twist curvature $F^S_{ij} \in \mathrm{End}_{\mathrm{Cl}}(S_q) = \mathrm{End}(S_q)_{\le 0}$, which is treated on exercise sheet 13. Putting this into the right hand side of (46) and comparing coefficients gives

$$t_\alpha = \frac{-1}{|\alpha|+1}\sum_{j:\ \alpha_j>0}\left[(R^S_{ij})_\alpha + (F^S_{ij})_\alpha\right] s_0.$$

Hence $t_0 = 0$ and t_α is in filtration $\le k+2$. This implies that $\nabla_X s \sim \sum t_\alpha x^\alpha$ is in filtration $\le k+1$, our first claim. To compute the symbol, we may work in F_{k+1}/F_k. Neglecting terms of filtration $\le k$, we see that $t_\alpha x^\alpha$, having filtration $\le k+2-|\alpha|$, vanishes unless $|\alpha| = 1$. Hence

$$\sigma_{k+1}(\nabla_X s) = \sum_j \sigma^{\mathrm{Cl}}_{k+2}(t_j)x^j = \sum_j \sigma_2^{\mathrm{Cl}}(R^S_{ij})\sigma_k^{\mathrm{Cl}}(s_0)x^j = \sum_j \frac{1}{4}R_{ij}\sigma_k^{\mathrm{Cl}}(s_0)x^j \qquad (47)$$

This clearly equals

$$\sigma_1^{\mathcal{G}}(\nabla_X)\sigma_k(s) = \left(\partial_i + \frac{1}{4}R_{ij}x^j\right)\sigma_k^{\mathrm{Cl}}(s_0).$$

This completes the proof in case s in synchronous. We now consider the general case. By the Leibniz rule for the covariant derivative we have[6] (see the following lemma for justification)

$$\nabla_X s \sim \sum_\alpha \left(\nabla_X(s_\alpha)x^\alpha + \alpha_i s_\alpha x^{\alpha - e_i}\right)$$

[5] We define the *twist curvature* F^S by $K^S = R^S + F^S$ for the *Riemannian endomorphism* given by $R^S(X,Y) = \frac{1}{2}\sum_{k<l} c(e_k)c(e_l)g(R(X,Y)e_k,e_l) \in \mathrm{End}(S)$. The twist curvature is a Cl-linear section of $\Omega^2(\mathrm{End}(S))$ (exercise sheet 13).

[6] We set $x^\alpha = 0$ if some $\alpha_i < 0$.

By the case we have already considered, $\nabla_X(s_\alpha)$ has canonical filtration $\leq k + |\alpha| + 1$. Hence both summands in the expansion have canonical filtration $\leq k+1$. For the symbol we get

$$\sigma_{k+1}(\nabla_X s) = \sum_\alpha \left[\sigma_{k+|\alpha|+1}(\nabla_X(s_\alpha))x^\alpha + \alpha_i \sigma_{k+|\alpha|}(s_\alpha)x^{\alpha - e_i}\right]$$

which agrees with (using the case already proven for s_α)

$$\sigma_1(\nabla_X)\sigma_k(s) = \sum_\alpha \left(\partial_i + \frac{1}{4}R_{ij}x^j\right)\sigma^{\mathrm{Cl}}_{k+|\alpha|}(s_\alpha)x^\alpha. \qquad \square$$

Lemma 170. *Let $s \in C^\infty(S \otimes S_q^*)$ have Taylor expansion $s \sim \sum s_\alpha x^\alpha$ at q. Then*

$$\nabla_X s \sim \sum_\alpha [\nabla_X(\tilde{s}_\alpha)x^\alpha + s_\alpha X(x^\alpha)]$$

where $\tilde{s}_\alpha$ is the synchronous extension of $s_\alpha \in \mathrm{End}(S_q)$ ($\nabla_X(\tilde{s}_\alpha)|_q$ is zero, but higher terms appear in the Taylor expansion. In the formula $\nabla_X(\tilde{s}_\alpha)$ denotes this Taylor expansion).

Proof. Let $\tau(x)$ denote parallel transport from q to x. Assuming the Taylor expansion $s(x) = \sum \tau(x)s_\alpha x^\alpha$ converges, the Leibniz rule shows

$$\nabla_X s = \sum_\alpha \nabla_X(\tau s_\alpha)x^\alpha + s_\alpha X(x^\alpha).$$

Finally, note that $\tilde{s}_\alpha(x) = \tau(x)s_\alpha$. $\square$

6.3.4 Reduction to the Mehler Formula

Recall the Getzler filtration on $\mathcal{D}^{\mathcal{G}}(S)$ and the canonical filtration on smoothing operators $C^\infty(S \boxtimes S^*)$. For these we defined symbol maps $\sigma^{\mathcal{G}}$ and σ which are compatible with the module structure of $C^\infty(S \boxtimes S^*)$ over $\mathcal{D}^{\mathcal{G}}(S)$. The module structure on the ranges of the symbol maps was defined in at the beginning of Section 6.3.3. This situation may be depicted as follows.

$$\begin{tikzcd}
\mathcal{D}^{\mathcal{G}}(S)_{\leq m} \arrow[r, bend left, "\text{composition}"] \arrow[dd, "\sigma^{\mathcal{G}}"'] & C^\infty(S \boxtimes S^*)_{\leq k} \arrow[d, "\text{Taylor}"] \\
 & C^\infty(\mathbb{C}[\![T^*M]\!] \otimes \mathrm{Cl}(TM) \otimes \mathrm{End}_{\mathrm{Cl}}(S))_{\leq k} \arrow[d, "\sigma"] \\
C^\infty(\mathfrak{P}(TM) \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))^m \arrow[r, bend left, "\text{module action}"] & C^\infty(\mathbb{C}[T^*M] \otimes \Lambda^*(T^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S))^k
\end{tikzcd}$$

Recall also that we studied the heat kernel k_t, which is characterized by

$$\left(\frac{\partial}{\partial t} + D_p^2\right)k_t(-,q) = 0, \qquad (\forall q \in M),$$

by means of an asymptotic expansion $k_t(-,q) = h_t(-,q)\left(\sum_{j=0}^\infty t^j u_j\right)$ where $u_j = \Theta_j(-,q)$. The main result of Section 5.3 was that the $u_j \in C^\infty(S \otimes S_q^*)$ may be determined recursively as solutions of the differential equation

$$\nabla_{\partial/\partial r}(r^j g^{1/4}u_j) = -r^{j-1}g^{1/4}D^2 u_{j-1}, \qquad u_{-1} = 0, \qquad u_0(q) = \mathrm{id}_{S_q}. \tag{48}$$

(see the proof of Lemma 130.) Of course for the Index Theorem, Equation (36), one does not need to know every u_j, it suffices to understand the Getzler symbol of $u_{n/2}$. By the remarkable design of the Getzler symbols, the symbols $\sigma^{\mathcal{G}}(u_j)$ are recursively determined by simpler equations, which we will derive from (48), that may be solved explicitly (Mehler formula).

Proposition 171. *The square of the Dirac operator $D^2 \in \mathcal{D}^{\mathcal{G}}(S)_{\leq 2}$ has Getzler filtration ≤ 2 and symbol at $q \in M$*

$$\sigma^{\mathcal{G}}(D^2)|_q = -\sum_i \left(\left.\frac{\partial}{\partial x^i}\right|_q + \frac{1}{4}\sum_j R_{ij}(q)x^j \right)^2 + F^S(q)$$

Here, $F^S(q)$ denotes the twist curvature at q (see page 69).

Here R_{ij} is computed with help of an orthonormal frame ($e_i = \frac{\partial}{\partial x^i}$) of T_qM and $(x^i) = (dx^i)$ is the dual frame on T_q^*M.

Proof. Putting the definition of twist curvature into the Weitzenböck formula (Theorem 33), one gets

$$D^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}_g + F.$$

Here, $F = \sum_{k<l} c(e_k)c(e_k)F^S(e_k, e_l)$ has Getzler filtration ≤ 2. Combined with

$$\nabla^*\nabla = -\sum_{i,j,k} g^{jk}\left(\nabla_j\nabla_k - \Gamma^i_{jk}\nabla_i\right)$$

it follows that D^2 has Getzler filtration ≤ 2. For symbols we have

$$\begin{aligned}
\sigma_2^{\mathcal{G}}(\nabla^*\nabla)_q &= \sigma_2(-\sum_i \nabla_i\nabla_i) = -\sum_i \left(\frac{\partial}{\partial x^i} + \frac{1}{4}R_{ij}(q)x^j\right)^2 \\
\sigma_2^{\mathcal{G}}(\mathrm{scal}_g) &= 0 \\
\sigma_2^{\mathcal{G}}(F)|_q &= \sum_{k<l} e_k e_k F^S(e_k, e_l)|_q = F^S(q).
\end{aligned}$$

□

For a function f and section s we have the composition formula

$$\sigma_k(fs) = \sum_{i+j=k} \sigma_i(f)\sigma_j(s). \tag{49}$$

In particular, if s has filtration $\leq k$, then

$$\sigma_k(fs) = f\sigma_k(s)$$

since f has filtration ≤ 0 and s has no components $> k$.

Proposition 172. *The (u_j) are in filtration $\leq 2j$. Let $v_j = \sigma_{2j}(u_j) \in \mathbb{C}[\![T_q^*M]\!] \otimes \Lambda^*(T_q^*M) \otimes \mathrm{End}_{\mathrm{Cl}}(S_q)$. Then*

$$jv_j + \sigma_0^{\mathcal{G}}(\nabla_{r\partial_r})v_j = -\sigma_2^{\mathcal{G}}(D^2)v_{j-1}, \tag{50}$$

where we set $\sigma_0^{\mathcal{G}}(\nabla_{r\partial_r}) := x^i\left(\frac{\partial}{\partial x^i} + \frac{1}{4}R_{ij}x^j\right)$.

We find the corresponding formula on the first part of p. 162 in [Roe] misleading. Roughly speaking the operator $\frac{\partial}{\partial r}$ appearing on the left of that equation must be replaced by the the Getzler symbol of this operator, which involves a curvature term. Fortunately, this is consistent with the proof of Corollary 173 below so that the the "symbol heat equation" for W derived in [Roe] holds.

Proof. We rewrite (48) as

$$jg^{1/4}u_j + \nabla_{r\partial_r}(g^{1/4}u_j) + g^{1/4}D^2u_{j-1} = 0$$

for all $r \neq 0$ and the radial vector field $\partial_r = \frac{1}{r}x^j\partial_j$, using normal coordinates at q. Hence $Y = r\partial_r$ is a smooth vector field which vanishes at q. Since by induction D^2u_{j-1} has canonical filtration $\leq 2j$ (by Proposition 171 and Theorem 165) it follows from Equation (27) that u_j has canonical filtration $\leq 2j$ at q. Applying the canonical symbol map at q gives

$$j\sigma_{2j}(u_j) + \sigma_{2j}(\nabla_{r\partial_r}g^{1/4}u_j) + \sigma_{2j}(D^2u_{j-1}) = 0.$$

Note that Lemma 168 does not apply to the middle term since u_j does not have filtration $\leq 2j-1$. It equals

$$\sigma_{2j}(\nabla_{r\partial_r}(g^{1/4}u_j)) = \sigma_{2j}(\nabla_{r\partial_r}(g^{1/4})\cdot u_j) + \sigma_{2j}(g^{1/4}\cdot\nabla_{r\partial_r}(u_j)) = \sigma_{2j}(\nabla_{r\partial_r}u_j),$$

applying Equation (49) because of the facts that $\nabla_{r\partial_r}u_j = x^i\nabla_i u_j$ has filtration $\leq 2j$ and

$$r\frac{\partial}{\partial r}g^{1/4} = r\frac{\partial}{\partial r}\big(1 - \frac{1}{24}\sum_{k,l}\mathrm{Ric}_{k,l}x^kx^l + \cdots\big) \quad \text{has filtration} \leq -1.$$

We now calculate this middle term $\sigma_{2j}(\nabla_{r\partial_r}u_j)$ by hand. Let $u_j \sim \sum_\alpha u_\alpha x^\alpha$ for $u_\alpha \in \mathrm{End}(S_q)$. Then from $Yx^\alpha = |\alpha|x^\alpha$ and Lemma 170 we get

$$\sigma_{2j}(\nabla_{r\partial_r}u_j) = \sum_\alpha \sigma_{2j+|\alpha|}(\nabla_{r\partial_r}\tilde{u}_\alpha)x^\alpha + \sigma_{2j+|\alpha|}(u_\alpha)|\alpha|x^\alpha.$$

Symbols of sections of the kind $\nabla_{r\partial_r}(\tilde{u}_\alpha)$ have be calculated in the proof of Lemma 168 **3.** in (47):

$$\sigma_{2j+|\alpha|}(\nabla_{r\partial_r}\tilde{u}_\alpha) = \frac{1}{4}R_{ij}\sigma_k^{\mathrm{Cl}}(u_\alpha)x^ix^j$$

Hence the middle term equals

$$\sigma_{2j}(\nabla_{r\partial_r}u_j) = \frac{1}{4}R_{ij}x^ix^j\sigma_{2j}(u_j) + x^i\frac{\partial}{\partial x^i}\sigma_{2j}(u_j) = x^i\left(\frac{\partial}{\partial x^i} + \frac{1}{4}R_{ij}x^j\right)v_j \qquad \square$$

The power series solutions $v_j = \sum a_Kx^K$ with $a_K \in \Lambda(T_q^*M)\otimes\mathrm{End}_{\mathrm{Cl}}(S_q)$ of the polynomial-coefficient differential equation of (50) are unique, given the initial condition $v_0 = \mathrm{id}_{S_q}$. For the proof, suppose by induction that $v_{j-1} = 0$. Then (50) is simply the difference equation

$$(j + |K|)a_K + \frac{1}{4}\sum_{i,j}R_{ij}a_{K-e_i-e_j} = 0,$$

where we set $a_K = 0$ if some component of $K = (k_1, \ldots, k_n)$ is negative. Another induction shows that $a_K = 0$. Hence $v_j = 0$.

Corollary 173. *For $t > 0$ let $W_t = h_t(v_0 + tv_1 + \cdots + t^{n/2}v_{n/2}) \in \mathbb{C}[T_q^*M]\otimes\Lambda^*(T_q^*M)\otimes\mathrm{End}_{\mathrm{Cl}}(S_q)$, where we set*[7] *$h_t(x) = (4\pi t)^{-n/2}\exp(-\|x\|^2/4t)$ and $v_j = \sigma_{2j}(u_j)|_q$. Then*

$$\frac{\partial W}{\partial t} + \sigma_2(D^2)|_qW_t = 0. \tag{51}$$

Let $w_j \in \mathbb{C}[T_q^*M]\otimes\Lambda^*(T_q^*M)\otimes\mathrm{End}_{\mathrm{Cl}}(S_q)$ have degree $2j$ and suppose $h_t(w_0 + tw_1 + \cdots + t^{n/2}w_{n/2})$ solves (51) with $w_0 = \mathrm{id}_{S_q}$. Then $w_j = \sigma_{2j}(u_j)|_q$. Hence the solutions of (51) of this form with given initial condition $w_0 = \mathrm{id}_{S_q}$ are unique.

[7] In the coordinates of T_q^*M induced by normal coordinates at q

Proof. It remains only to see that solutions of (51) of the type $h_t(v_0 + tv_1 + \cdots + t^{n/2}v_{n/2})$ correspond exactly to solutions of the system of equations (50). An explicit computation shows the following analogue of Lemma 130

$$\sigma_2^{\mathcal{G}}(D^2)(h_t \cdot s) = h_t \cdot \sigma_2^{\mathcal{G}}(D^2)(s) + (\Delta h_t) \cdot s + \frac{h_t}{t}\sigma_0^{\mathcal{G}}(\nabla_{r\partial_r})(s).$$

Recall also

$$\Delta h_t = \left(\frac{n}{2t} - \frac{r^2}{4t^2}\right) h_t, \qquad \frac{\partial h}{\partial t} = \left(\frac{-n}{2t} + \frac{r^2}{4t^2}\right) h_t$$

Putting the last two equations into the left-hand side of (51) and dividing by h_t gives

$$\begin{aligned}\left(\frac{n}{2t} - \frac{r^2}{4t^2}\right) v_j t^j + (j+1)v_{j+1}t^j + \left(\frac{n}{2t} - \frac{r^2}{4t^2} v_j t^j\right) + \sigma_0^{\mathcal{G}}(\nabla_{r\partial_r})v_{j+1}t^j \\ = \left((j+1)v_{j+1} + \sigma_2(D^2)(v_j) + \sigma_0^{\mathcal{G}}(\nabla_{r\partial_r})(v_{j+1})\right) t^j\end{aligned}$$

This power series is zero precisely when (50) holds. □

6.4 The Mehler Formula

Proposition 174 (Mehler Formula)**.** *Let $R \in \mathbb{C}^{n\times n}$ be a complex skew-symmetric matrix $R^T = -R$, let $F \in \mathbb{C}$. Suppose $n = 2m$ is even. The differential equation*

$$\frac{\partial w}{\partial t} - \sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i} + \frac{1}{4}R_{ij}x^j\right)^2 w_t + Fw_t = 0 \tag{52}$$

has the (obviously analytic in R) solution

$$w_t^R(x) = (4\pi t)^{-n/2} \det{}^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right) \exp\left(-\frac{1}{4t}\left\langle \frac{tR}{2}\coth\frac{tR}{2}x, x\right\rangle\right)\exp(-tF). \tag{53}$$

Remark 175. *If R is not invertible, the expression* $\det^{1/2}$ *is set to zero. Else the generalized eigenvalues*[8] *of R occur in pairs $\pm\lambda_1, \ldots, \pm\lambda_m$ and give rise to double eigenvalues of the symmetric matrix $\frac{tR/2}{\sinh tR/2}$, whose determinant is therefore $\prod_{i=1}^m \left(\frac{t\lambda_i/2}{\sinh t\lambda_i/2}\right)^2$. We define*

$$\det{}^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right) = \prod_{i=1}^{m} \frac{t\lambda_i/2}{\sinh t\lambda_i/2}.$$

Alternatively, one may use the Jordan canonical form of R to define $f(R)$ for any power series $f(z)$. The expression $\coth\frac{tR}{2}$ is defined in this way using the power series $\coth(tz/2)$. Then one can also define $\det^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right) = \det(f(R))$ for $f(z) = \sqrt{\frac{tz/2}{\sinh tz/2}}$, the unique square-root of the power series with constant term 1.

Proof. In principle, we need only put (53) into (52). To make the calculation more manageable, we do a series of reductions. Since R is skew-symmetric we find a unitary matrix U so that

$$U^T \cdot R \cdot U = \begin{pmatrix} 0 & -\lambda_1 & & & \\ \lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\lambda_m \\ & & & \lambda_m & 0 \end{pmatrix}.$$

[8] The zeros of $\det(R - X \cdot \mathbb{E}_n)$

In case $R = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ and $F = 0$, formula (53) simplifies to

$$w_t^\theta(x) := (4\pi t)^{-1} \left(\frac{t\vartheta}{\sinh(t\vartheta)}\right) \exp\big(-\frac{1}{4}\vartheta|x|^2\coth(t\vartheta)\big), \quad \vartheta = i\theta/2.$$

The function w_t^R may be expanded in terms of these as a product

$$w_t^R(Ux) = w_t^{\lambda_1}(x_1, x_2)\cdots w_t^{\lambda_n}(x_{n-1}, x_n)\exp(-tF).$$

Since U is linear, an easy application of the Leibniz and chain rule shows that w_t^R solves (52) in case each w_t^θ solves the corresponding 2-dimensional equation, which we rewrite as

$$\begin{aligned} \frac{\partial w^\theta}{\partial t} &= \left(\frac{\partial}{\partial x} - \frac{\theta y}{4}\right)^2 w_t^\theta + \left(\frac{\partial}{\partial y} + \frac{\theta x}{4}\right)^2 w_t^\theta \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_t^\theta + \frac{1}{16}\theta^2(x^2+y^2)w_t^\theta + \underbrace{\frac{1}{2}\theta\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)w_t^\theta}_{=0}. \end{aligned}$$

The last summand vanishes since w_t^θ is obviously rotationally symmetric, so it remains to check

$$\frac{\partial w^\theta}{\partial t} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_t^\theta + \frac{1}{16}\theta^2(x^2+y^2)w_t^\theta. \tag{54}$$

Clearly, $w_t^\theta(x,y) = w_t^\theta(x)\cdot w_t^\theta(y)$ is a product of functions of one variable, given by

$$w_t^\theta(x) = (4\pi t)^{-1/2}\left(\frac{t\vartheta}{\sinh(t\vartheta)}\right)^{1/2} \exp\big(-\frac{1}{4}\vartheta x^2\coth(t\vartheta)\big), \quad \vartheta = i\theta/2.$$

To prove (54), it suffices to check the following for this one-dimensional function:

$$\frac{\partial w^\theta}{\partial t} = \frac{\partial^2 w^\theta}{\partial x^2} - \frac{1}{4}\vartheta^2 x^2 w_t^\theta.$$

We calculate:

$$\begin{aligned} \frac{\partial w}{\partial t} &= \left(-\frac{1}{2t} + \frac{1 - t\vartheta\coth(t\vartheta)}{2t} - \frac{1}{4}\vartheta^2x^2(1-\coth(t\vartheta)^2)\right) w_t^\theta \\ &= \left(-\frac{1}{2}\vartheta\coth(t\vartheta) - \frac{1}{4}\vartheta^2x^2(1-\coth(t\vartheta)^2)\right) w_t^\theta \\ \frac{\partial w}{\partial x} &= -\frac{1}{2}\vartheta x\coth(t\vartheta)\cdot w_t^\theta \\ \frac{\partial^2 w}{\partial x^2} &= -\frac{1}{2}\vartheta\coth(t\vartheta)w_t^\theta + \frac{1}{4}\vartheta^2x^2\coth^2(t\vartheta)w_t^\theta \end{aligned}$$

□

Inserting tangent vectors $X, Y \in T_qM$ into (51), Proposition 171 shows the resulting differential equation to be of the form (52). The solution (53) is a polynomial in t since for the curvature 2-form $R^j = 0$ for $j > n/2$. Hence the uniqueness statement in Proposition 51 applies:

Corollary 176. *For each $q \in M$ let W_t be defined as in Corollary 173. Then*

$$W_t = (4\pi t)^{-n/2}\det{}^{1/2}\left(\frac{tR/2}{\sinh tR/2}\right)\exp\left(-\frac{1}{4t}\left\langle\frac{tR}{2}\coth\frac{tR}{2}x, x\right\rangle\right)\exp(-tF^S(q)),$$

for the curvature matrix R at q and the twist curvature $F^S(q)$

6.5 Genera

6.5.1 Complex Vector Bundles

Let $f = 1 + \sum_{k=1}^{\infty} a_k X^k \in \mathbb{C}[\![X]\!]$ be a normalized formal power series with complex coefficients a_i.

For a complex vector bundle $E \to M$ with inner product and compatible connection, we denote the curvature 2-form by $R^{\nabla} \in \Omega^2(M; \operatorname{End} E)$.

Definition 177. *The* Chern f-genus *is defined as*

$$K_f(E, \nabla) = \det f\left(\frac{i}{2\pi} R^{\nabla}\right) \in \Omega^{ev}(M; \mathbb{C}).$$

Here, the (nilpotent) curvature form is formally inserted into the formal power series f. Thus[9]

$$f\left(\frac{i}{2\pi} R^{\nabla}\right) = 1 + \sum_{k=1}^{\infty} a_k \left(\frac{i}{2\pi}\right)^k \underbrace{R^{\nabla} \wedge \cdots \wedge R^{\nabla}}_{k \text{ times}} \in \Omega^{ev}(M; \operatorname{End}(E)),$$

which is then post-composed with the determinant $\operatorname{End}(E) \to \mathbb{C}$.

The proof of the following proposition can be found in the appendix of the book Characteristic Classes by Milnor-Stasheff [MS]. It is based on the Bianchi identity.

Proposition 178. *The differential form $K_f(E, \nabla)$ is closed. Its de Rham cohomology class is independent of the chosen metric and connection on E.*

Definition 179. *The* Chern polynomials $c_k \colon \mathbb{C}^{n \times n} \to \mathbb{C}$ *are defined as the k-homogeneous components in*

$$\sum_{k=0}^{n} c_k(R) t^k = \det\left(1 + tR\right).$$

Hence $c_0(R) = 1, c_1(R) = \operatorname{tr} R$, and $c_n(R) = \det R$. The k-th Chern form *of the complex vector bundle E is*

$$c_k(E, \nabla) = c_k\left(\frac{i}{2\pi} R^{\nabla}\right).$$

The de Rham cohomology classes are called Chern classes. *It can be shown [MS] that they represent classes in the integral cohomology $[c_k(E, \nabla)] \in H^{2k}(M; \mathbb{Z})$, the k-th* Chern class *of E.*

More generally, $c_k(R)$ is the k-th elementary symmetric polynomial $\sigma_k(\lambda_1, \ldots, \lambda_n)$ in the eigenvalues of R. Returning to Chern classes for a complex vector bundle E, pure algebra allows us to adjoin n elements $x_1, \ldots, x_n$ to the de Rham cohomology algebra so that we may write

$$c_k(E, \nabla) = \sigma_k(x_1, \ldots, x_n).$$

It is not hard to see then that we may rewrite the Chern f-genus as

$$[K_f(E, \nabla)] = \prod_{j=1}^{n} f(x_j) =: K_f(c_1, c_2, \ldots)$$

Since the product is symmetric in the x_j, it may be expanded in terms of the elementary symmetric polynomials $c_k(E, \nabla)$ which defines K_f on the right.

[9]Note that only finitely many summands are non-zero

Definition 180. *The* Chern character *of E is*

$$\mathrm{ch}(E) = \mathrm{tr}\left(\exp\left(\frac{i}{2\pi}R^\nabla\right)\right) \in \Omega^{ev}(M;\mathbb{C}).$$

This defines a closed differential form and hence a de Rham cohomology class. Using the formal variables above the Chern character may be rewritten as

$$[\mathrm{ch}(E)] = \sum_{i=1}^n e^{x_i} = \dim E + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots$$

Suppose that $S \to M$ is a complex Dirac bundle. Recall from (38) the decomposition

$$\mathrm{End}_{\mathbb{C}}(S_q) = \mathrm{Cl}(n) \otimes \mathrm{End}_{\mathrm{Cl}}(S_q), \qquad \mathrm{End}_{\mathrm{Cl}}(S_q) = \mathrm{End}_{\mathbb{C}}(V)$$

where $S_q = \Delta \otimes V$ in the isotypical decomposition. Decomposing an endomorphism $F = c \otimes f$ in this way defined, as we recall, the *relative trace* $\mathrm{tr}^{S/\Delta} F = \mathrm{tr}(f)$ (this is related to the super trace of F by Equation 41). The curvature R^∇ of S is a 2-form with values in $\mathrm{End}_{\mathbb{C}}(S)$. The twist curvature F^S is a 2-form with values in $\mathrm{End}_{\mathrm{Cl}}(S_q)$. In this special situation we define:

Definition 181. $\mathrm{ch}(S/\Delta) = \mathrm{tr}^{S/\Delta} \exp\left(\frac{i}{2\pi}F^S\right)$

For example, if $S = \Delta^S \otimes V$ has the tensor product connection ($\Delta^S = P_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}} \Delta$ is the spinor bundle with its connection inherited from M and V is another vector bundle with inner product and compatible connection) then $\mathrm{ch}(S/\Delta) = \mathrm{ch}(V)$.

6.5.2 Real Vector Bundles

Let $E \to M$ be a real vector bundle with inner product and compatible connection ∇. Let $g(z) = 1 + \sum_{k=1}^\infty \alpha_i z^i$ be a normalized formal power series. Define

$$f(z) = \sqrt{g(z^2)} = 1 + \sum_{i=1}^\infty \beta_{2i} z^{2i}$$

where the root has been chosen so that f is normalized.

Definition 182. *The* Pontrjagin g-genus *is*[10]

$$K_g(E, \nabla) = \det\left(f\left(\frac{i}{2\pi}R^\nabla\right)\right).$$

Similarly as for complex bundles we may introduce new formal variable $y_1, \ldots, y_n$ (the rank of E is n) for which

$$[K_g(E,\nabla)] = \prod_{i=1}^n g(y_i) =: K_g(p_1, p_2, \ldots), \qquad p_i = \sigma_i(y_1, \ldots, y_n).$$

Again it follows from the theory of characteristic classes [MS] that $p_k \in H^{4k}(E;\mathbb{Z})$ can be identified with the k-th *Pontrjagin class* of E.

Example 183. *1. The $\hat{A}$-series is $g^{\hat{A}}(z) = \frac{\sqrt{z}/2}{\sinh \sqrt{z}/2} = 1 - \frac{1}{24}z + \frac{7}{5760}z^2 \cdots$. The corresponding Pontrjagin genus is called the* $\hat{A}$-genus *(or $\hat{A}$-class). We have*

$$K_{g^{\hat{A}}}(p_1, p_2, \ldots) = g(y_1)\cdots g(y_n) = 1 - \frac{1}{24}(y_1 + \cdots + y_n) + \frac{7}{5760}(y_1^2 + \cdots + y_n^2) + \frac{1}{24^2}(y_1y_2 + y_1y_3 + \cdots)$$
$$= 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots$$

[10] This is just the Chern f-genus of $E \otimes \mathbb{C}$

2. *The L-series is* $g^L(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}} = 1 + \frac{1}{3}z - \frac{1}{45}z^2 + \cdots$*. The Pontrjagin genus for L is called the L-*genus *or L-class. Then*

$$K_{g^L}(p_1, p_2, \ldots) = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots$$

6.6 Proof of the Index Theorem

After 77 pages of tedious preparation, we happily arrive at a complete proof of the main theorem:

Theorem 184 (Atiyah-Singer Index Theorem)**.** *Let $S \to M$ be a Dirac bundle over a closed oriented Riemannian manifold $M^{n=2m}$ with the canonical grading. Then we may calculate the index as*

$$\mathrm{ind}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}(S/\Delta).$$

Proof. In Theorem 141 we found the following expression for the index:

$$\mathrm{ind}(D) = \frac{1}{(4\pi)^{n/2}} \int_M \mathrm{tr}_S\, \Theta_{n/2}(q,q) d\mathrm{vol}(q).$$

Recall that we write $\Theta_j(p,q) = u_j^{(q)}(p)$. By (41) the super trace of $\sum c_I e_I \otimes \phi^I = u_{n/2}^{(q)}(q) \in \mathrm{Cl}(n) \otimes \mathrm{End}_{\mathrm{Cl}}(S_q)$ is $(-2i)^m c_{\{1,\ldots,n\}} \mathrm{tr}^{\,S/\Delta} \phi^{\{1,\ldots,n\}} d\mathrm{vol}(q) = (-2i)^m \mathrm{tr}^{\,S/\Delta} \sigma_n(u_{n/2})$ (the Getzler symbol $v_{n/2} = \sigma_n(u_{n/2})$ exactly picks out this component). Hence

$$\begin{aligned}
\mathrm{ind}(D) &= \frac{1}{(2\pi i)^m} \int_M \mathrm{tr}^{\,S/\Delta} v_{n/2}^{(q)}(q) \\
&= \frac{1}{(2\pi i)^m} \int_M \mathrm{tr}^{\,S/\Delta} \left(v_0^{(q)}(q) + \cdots + v_{n/2}^{(q)}(q) \right) \\
&= \frac{1}{(2\pi i)^m} \int_M \mathrm{tr}^{\,S/\Delta} W_{t=1}^q(q) \\
&= \frac{1}{(2\pi i)^m} \int_M \det{}^{1/2} \left(\frac{R/2}{\sinh R/2} \right) \mathrm{tr}^{\,S/\Delta}(\exp(-F^S)) \\
&= \frac{1}{(2\pi i)^m} \int_M \det{}^{1/2} \left(\frac{-R/2}{\sinh -R/2} \right) \mathrm{tr}^{\,S/\Delta}(\exp(-F^S)) \\
&= \int_M \det{}^{1/2} \left(\frac{\frac{i}{2\pi} R/2}{\sinh \frac{i}{2\pi} R/2} \right) \mathrm{tr}^{\,S/\Delta}(\exp(\frac{i}{2\pi} F^S)) \\
&= \int_M \hat{A}(TM) \wedge \mathrm{ch}(S/\Delta)
\end{aligned}$$

□

6.7 First Applications of the Index Theorem

Example 185. *Let $M^{n=2m}$ be a spin manifold and consider the Dirac bundle $S^\Delta = P_{\mathrm{Spin}} \times_{\mathrm{Spin}} \Delta$ for the unique irreducible representation Δ of* Spin *(see Section* 3.5.1*), along with the canonical grading induced by $\Delta = \Delta^+ \oplus \Delta^-$. As calculated on exercise sheet 13, the twisting curvature $F^S = 0$ vanishes in this case. Therefore, using Example* 183 *we get*

$$\mathrm{ind}(D) = \int_M \hat{A}(TM) = \int_M \left(1 - \frac{p_1}{24} + \frac{1}{5760}(-4p_2 + 7p_1^2) + \cdots \right).$$

In particular, for the spinor Dirac operator we have $\mathrm{ind}(D) = 0$ *unless n is a multiple of four.*

Theorem 186 (Lichnerowicz)**.** *Let M be closed spin manifold with $\hat{A}(M) \neq 0$. Then there exists no metric g on M with positive scalar curvature* $\mathrm{scal}_g > 0$.

Proof. Recall the Weitzenböck formula $D^2 = \Delta + \frac{1}{4}\mathrm{scal}_g$ in Theorem 55 for the spinor Dirac operator. Suppose $\mathrm{scal}_g > 0$. To prove the theorem we must show $\mathrm{ind}(D) = 0$, by Example 185. For this we prove that $\ker(D_+) = 0$; a similar argument shows $\ker(D_-) = 0$. Let $Ds = 0$ for $s \in C^\infty(S_+)$. Then

$$\begin{aligned} 0 = \langle D^2 s, s\rangle &= \langle \Delta s, s\rangle + \frac{1}{4}\mathrm{scal}_g\langle s, s\rangle \\ &= \langle \nabla s, \nabla s\rangle + \frac{1}{4}\mathrm{scal}_g\langle s, s\rangle \end{aligned}$$

is a sum of non-negative numbers. It follows that these must both be zero, so $\|s\|^2 = 0$ and $s = 0$. This proves $\ker(D_+) = 0$. ☐

Example 187. *For a K3-surface, $\hat{A}(M) \neq 0$ (this is a certain 4-manifold for which specific constructions exist, but which usually defined in terms of its properties). The signature of a K3-surface may be computed from its Hodge diamond, using a famous theorem by Hodge. Hence $L(M) = -16$ and $\hat{A}(M) = -\frac{1}{8}\cdot L(M) = 2$. Here, the relationship between $L(M)$ and $\hat{A}(M)$ for 4-dimensional manifolds follows from Example* 183. *Note that the K3-surface is spin.*

The spin condition is essential in Theorem 186. For example, $\hat{A}(\mathbb{C}P^2) = -1/8$, but $\mathbb{C}P^2$ has a metric of positive scalar curvature (the Fubini-Study metric). In particular, $\mathbb{C}P^2$ is not spin.

Theorem 188 (Hirzebruch Signature Theorem)**.** *Let $M^{n=2m}$ be an oriented closed manifold. Then*

$$\mathrm{signature}(M) = \int_M L(M).$$

(*recall here the Pontrjagin L-genus from Example* 183.)

Proof. We have already seen in Examples 135 and 138 that the complexified bundle of exterior forms $S = \Lambda^*(T^*M)\otimes\mathbb{C}$ with the grading given essentially by the Hodge operator leads to the signature operator $D = d + d^*$. The twist curvature can be computed to be given by the Pontrjagin f-class for $f(z) = 2^m\cosh(\sqrt{z}/2)$. Hence the signature itself is the Pontrjagin genus for $2^m\frac{\sqrt{z}/2}{\sinh\sqrt{z}/2}\cdot\cosh(\sqrt{z}/2)$, which is the same as the genus for $\frac{\sqrt{z}}{\tanh\sqrt{z}}$. ☐

Example 189. *Since $L(TM) = 1 + \frac{p_1}{3} + \frac{1}{45}(7p_2 - p_1^2) + \cdots$, for 4-manifolds M^4 we get*

$$\mathrm{signature}(M^4) = \frac{p_1(M)}{3}.$$

Similarly, for M spin $\mathrm{ind}(D) = \hat{A}(M) = -\frac{p_1}{24}$.

The signature and index are integers. This leads to divisibility theorems for characteristic numbers. For example, the signature theorem implies that $p_1(M)$ is always a multiple of 3!

Theorem 190 (Rokhlin)**.** *If M^4 is a 4-dimensional closed spin manifold, then $\hat{A}(M)$ is even. In particular, the signature of a closed spin 4-manifold is divisible by* 16.

Proof. The Clifford algebra $\mathrm{Cl}(4)$ may be identified with $\mathbb{H}^{2\times 2}$ and the unique irreducible representation Δ of $\mathrm{Cl}(4)$ is then $\mathbb{H}^2$. The action is by usual matrix-vector multiplication. In particular, Δ is a quaternionic vector space. This quaternionic structure (from the right) is compatible with Clifford multiplication (from the left), so the associated spinor Dirac bundle is a quaternionic vector bundle. Moreover, the Dirac operator is equivariant under $\mathbb{H}$. In particular, the spaces $\ker(D_\pm)$ are quaternionic, so their complex dimension is a multiple of two. ☐

A famous application of the Atiyah-Singer index theorem in *complex geometry* is the theorem of *Hirzebruch-Riemann-Roch*. Some information can be found on p. 175 ff. in [Roe].

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